Gromov-Witten theory of elliptic orbifold projective lines

by

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To my grandma

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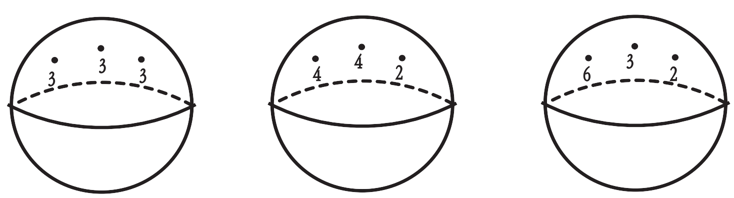
CHAPTER I

Introduction

Gromov-Witten theory is a mathematical theory originated from the string theory. It has been in the center of geometry and physics for the last twenty years. This thesis will focus on the Gromov-Witten theory of three types of elliptic orbifold projective lines, $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{4,4,2}$ and $\mathbb{P}^1_{6,3,2}$. They are all quotient spaces of elliptic curves in some weighted projective spaces under actions of finite groups. The underlying spaces are captured as follows, where the numbers show the order of the isotropy cyclic group at the orbifold points.



The elliptic curves have deep connections to singularity theory. In 2007, a new Gromov-Witten type theory was introduced for nondegenerate quasihomogeneous hypersurface singularities, by Fan, Jarvis and Ruan, based on a proposal by Witten. This is the so called FJRW theory. It is believed to be the counterpart of the Gromov-Witten theory in the so called Landau-Ginzburg model. The relationship between two theories is referred to as the Landau-Ginzburg/Calabi-Yau correspondence, a famous duality from physics. Landau-

Ginzburg/Calabi-Yau correspondence can be cast into the framework of the global mirror symmetry. Comparing to the more traditional mirror symmetry, global mirror symmetry emphasis the global aspect of mirror symmetry such as the analytic continuation of Gromov-Witten theory. It leads naturally to the modularity of Gromov-Witten generating function, a surprising and yet beautiful new perspective of the subject.

1.1 Landau-Ginzburg/Calabi-Yau correspondence

There is a great deal of interest recently in studying the so called Landau-Ginzburg /Calabi-Yau correspondence or LG/CY correspondence. Mathematically, the LG/CY correspondence is concerned with the equivalence of two mathematical theories originating from a quasihomogeneous polynomial of Calabi-Yau type. A polynomial $W : \mathbb{C}^N \to \mathbb{C}$ is quasihomogeneous if there is an N-tuple of rational numbers (weights) $(q_1, \cdots, q_N)$ such that for any $\lambda \in \mathbb{C}^*$,

$$W(\lambda^{q_1} X_1 \cdots, \lambda^{q_N} X_N) = \lambda W(X_1, \cdots, X_N).$$

We assume that W is nondegenerate in the sense that it defines an isolated singularity at the origin. W is called of *Calabi-Yau type* if

$$\sum_{i=1}^{N} q_i = 1.$$

Another piece of data is a subgroup G of the *maximal diagonal symmetry group* G_W, where

$$G_W := \left\{(\lambda_1, \cdots, \lambda_N) \in (\mathbb{C}^*)^N ; W(\lambda_1 X_1, \cdots, \lambda_N X_N) = W(X_1, \cdots, X_N)\right\}$$

G_W has a special element

$$J = (\exp(2\pi i q_1), \cdots, \exp(2\pi i q_N)).$$

We say $G \subset G_W$ is *admissible* if it contains J.

The geometric realization of a Calabi-Yau type quasi-homogeneous polynomial is that the equation $W = 0$ defines a Calabi-Yau hypersurface X_W in the weighted projective space $\mathbb{WP}(c_1, \cdots, c_N)$, where $q_i = c_i/d$ for a common denominator d. G acts naturally on X_W by multiplication on coordinates with the subgroup $\langle J \rangle$ acting trivially. We define a group

$$\widetilde{G} := G/\langle J \rangle.$$

Hence $\widetilde{G}$ acts faithfully on X_W. One side of the LG/CY-correspondence is the (orbifold) Gromov-Witten theory of the quotient of the Calabi-Yau hypersurface,

$$\mathcal{X}_{W,G} := (X_W = \{W = 0\}) / \widetilde{G}.$$

If $G = G_W$, we simply omit the group and denote the orbifold by $\mathcal{X}_W := \mathcal{X}_{W,G_W}$. In later chapters, our paper will focus only on the cases with $G = G_W$.

Gromov-Witten theory is now well-known in mathematics. It was first constructed for semi-Fano symplectic manifolds in [RT] and later in many other papers for various generalizations. Later on, a similar theory was constructed for orbifolds in symplectic setting [CheR1, CheR2]. For algebraic constructions, refer to [AbGV]. We also refer readers to [ALR] for more details about the Chen-Ruan cohomology and the references there. The main elements of the (orbifold) Gromov-Witten theory for the Calabi-Yau space $\mathcal{X}_{W,G}$ are summarized as follows

(1) There exista a state space: Chen-Ruan orbifold cohomology $H^*_{CR}(\mathcal{X}_{W,G})$;

(2) There are numerical invariants $\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n) \rangle^{\mathcal{X}_{W,G}}_{g,n,\beta}$ defined by a virtual counting of stable maps. Here, g is the genus of the source curve and β is the fundamental class of image of the stable maps. β is in the Mori cone of $\mathcal{X}$, i.e. $\beta \in NE(\mathcal{X})$. We assembles those invariants into a generating function $\mathcal{F}^{GW}_{g,\mathcal{X}_{W,G}}(q^\beta)$ in infinitely many variable indexed by a basis $\{\alpha_i\}$ of the state space, a variable z to keep track of the

sum of integers l_i and a Novikov variable q^β to keep track of β. One can further sum over genera to define the total ancestor potential function

$$\mathcal{A}^{GW}_{\mathcal{X}_{W,G}}(q^\beta) = \sum_{g\geq 0} \hbar^{2g-2}\mathcal{F}^{GW}_{g,\mathcal{X}_{W,G}}(q^\beta).$$

We should emphasize that $\mathcal{F}^{GW}_{g,\mathcal{X}_{W,G}}$ is only a formal power series;

(3) $\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n)\rangle_{g,n,\beta}$ satisfies a set of axioms referred as *cohomological field theory axioms* (see Chapter II for details).

The other side of the LG/CY-correspondence is the FJRW theory of the singularity (W, G) constructed by Fan, Jarvis and Ruan in a series of papers [FJR1, FJR2, FJR3], based on a proposal of Witten. The FJRW theory is very different from Gromov-Witten theory. However, it shares the same general structure with Gromov-Witten theory as an example of cohomological field theory. It has the following properties:

(1) A state space $\mathscr{H}^{FJRW}_{W,G}$(or $\mathscr{H}_{W,G}$ for short. See its definition in Chapter III);

(2) Numerical invariants by a virtual counting of solutions of the Witten equation, its generating functions $\mathcal{F}^{FJRW}_{g,W,G}(t), \mathcal{A}^{FJRW}_{W,G}(t)$ (t is a certain degree 2 variable playing the role of Kähler parameter);

(3) It satisfies the cohomological field theory axioms.

Motivated by physics, Yongbin Ruan has formulated a striking mathematical conjecture to relate the two theories. For more details, we refer to [Ru] and [ChiR3]. One of the goals of this paper is to prove this conjecture for elliptic orbifold $\mathbb{P}^1$. Ruan's conjecture is stated as follows.

Conjecture I.1. *Let W be a nondegenerate quasi-homogeneous polynomial of Calabi-Yau type and let G be an admissible group.*

(1) There is a graded vector space isomorphism

$$\mathscr{H}_{W,G}^{FJRW} \to H_{CR}^{*}\left(\mathcal{X}_{W,G}\right).$$

Hence, we can identify the two state spaces.

(2) There is a degree-preserving $\mathbb{C}[z, z^{-1}]$-valued linear symplectic isomorphism $\mathbb{U}_{W,G}$ of so-called Givental symplectic vector spaces and a choice of analytic continuation of Givental cones $\mathcal{L}_{W,G}^{FJRW}$ and $\mathcal{L}_{\mathcal{X}_{W,G}}^{GW}$ with respect to the Kähler parameter such that

$$\mathbb{U}_{W,G}\left(\mathcal{L}_{W,G}^{FJRW}\right) = \mathcal{L}_{\mathcal{X}_{W,G}}^{GW}.$$

(3) The total potential functions are related by quantization of $\mathbb{U}_{\mathrm{LG/CY}}$, up to a choice of analytic continuation. We simply denote by

$$\mathcal{A}_{\mathcal{X}_{W,G}}^{GW} \equiv \widehat{\mathbb{U}}_{W,G}\left(\mathcal{A}_{W,G}^{FJRW}\right).$$

Here $\equiv$ means two sides are equal modulo an analytic continuition.

Part (1) is called the *Cohomological LG/CY correspondence*, which has been verified in full generality by Chiodo-Ruan [ChiR2]. Part (2) is the genus-0 LG/CY correspondence, which has been verified by Chiodo-Ruan [ChiR1] for the quintic 3-fold, and by Chiodo-Iritani-Ruan [CIR] for all Fermat hypersurfaces and $G = \langle J \rangle$. The fisrt example of the conjecture for all genera is proved in [KS] and [MR], for three classes of orbifold $\mathbb{P}^1$, i.e. elliptic orbifold projective lines $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{4,4,2}$ and $\mathbb{P}^1_{6,3,2}$. Namely, there exist pairs (W, G), such that the Calabi-Yau sides of them are those elliptic orbifolds, and there exists an operator $\widehat{\mathbb{U}}_{W,G}$, to connect the generating functions from the FJRW side to the Calabi-Yau side.

Let us denote W by

$$W(\mathbf{X}) = \sum_{i=1}^{s} \prod_{j=1}^{N} X_j^{a_{ij}}, \quad \mathbf{X} = (X_1, \ldots, X_N)$$

We define its exponent matrix E_W by taking the exponents that appear in the monomials of W as entries, i.e. the (i,j)-th entry of E_W is a_{ij}. We say W is *invertible*, if $s = N$ and E_W is an invertible matrix.

According to Saito [Sa2], simple elliptic singularities are classified in three cases $E_{\mu-2}^{(1,1)}$, $\mu = 8, 9, 10$. Here μ is actually the dimension of the Jacobi algebra of W, as a vector space. However, as a polynomial, there are different choices of normal forms for each singularity. For example, the Fermat cubic singularity $W = X_1^3 + X_2^3 + X_3^3$ is of type $E_6^{(1,1)}$. We simply denote by $W \in E_6^{(1,1)}$. All simple elliptic singularities of the same type have isomorphic Jacobi algebras. However, their FJRW theory will be different. Here we list all invertible simple elliptic singularities.

Table 1.1: Invertible simple elliptic singularities

	$E_6^{(1,1)}$	$E_7^{(1,1)}$	$E_8^{(1,1)}$
Fermat	$X_1^3 + X_2^3 + X_3^3$	$X_1^4 + X_2^4 + X_3^2$	$X_1^6 + X_2^3 + X_3^2$
Fermat+Chain	$X_1^2X_2 + X_2^3 + X_3^3$	$X_1^3X_2 + X_2^4 + X_3^2$	$X_1^4X_2 + X_2^3 + X_3^2$
		$X_1^2X_2 + X_2^2 + X_3^4$	$X_1^3X_2 + X_2^2 + X_3^3$
Fermat+Loop	$X_1^2X_2 + X_1X_2^2 + X_3^3$	$X_1^3X_2 + X_1X_2^3 + X_3^2$	
Chain	$X_1^2X_2 + X_2^2X_3 + X_3^3$	$X_1^3X_2 + X_2^2X_3 + X_3^2$	
Loop	$X_1^2X_2 + X_2^2X_3 + X_1X_3^2$		

We recall that in [KreS], it is proved that an invertible polynomial is nondegenerate if and only if it can be written as a sum of the following three types:

1. Fermat: $W = X_1^{r_1} + \cdots + X_N^{r_N}$.
2. Loop: $W = X_1^{r_1}X_2 + X_2^{r_2}X_3 + \cdots + X_{N-1}^{r_{N-1}}X_N + X_N^{r_N}X_1$.
3. Chain: $W = X_1^{r_1}X_2 + X_2^{r_2}X_3 + \cdots + X_{N-1}^{r_{N-1}}X_N + X_N^{r_N}$.

It is not hard to see that if W is an invertible simple elliptic singularity (or ISES for short), then $\mathcal{X}_{W,G_W}$ is an elliptic orbifold $\mathbb{P}^1$. The explicit correspondence will be discussed later in global mirror symmetry section.

We say W is a *good invertible simple elliptic singularity* if W is a singularity in the Table 1.1 and W is not of the form $X_1^2X_2 + X_2^3 + X_3^3 \in E_6^{(1,1)}$, $X_1^2X_2 + X_1X_2^2 + X_3^3 \in E_6^{(1,1)}$, or $X_1^2 + X_2^2 + X_3^4 \in E_7^{(1,1)}$. Our main theorem of this thesis is

Theorem I.2. *[KS, MR, MS] Let W be a good invertible simple elliptic singularity, and G_W be its maximal diagonal symmetry group. Then the LG/CY correspondence of all genera holds true for the pair (W, G_W). More precisely, there exists an operator $\widehat{\mathbb{U}}_W$, such that*

$$\widehat{\mathbb{U}}_W\left(\mathscr{A}_W^{FJRW}\right) \equiv \mathscr{A}_{X_W}^{GW}. \tag{1.1}$$

This theorem is first proved in [KS, MR] for three special types of cubic polynomials, where $W = X_1^3+X_2^3+X_3^3$, $X_1^2X_2+X_2^2X_3+X_3^3$ and $X_1^2X_2+X_2^3+X_3^3$. Later, the statement is generalized to all other good cases in [MS]. Here, let us mention that three cubic singularities we listed above are all belong to $E_6^{(1,1)}$. However, the corresponding orbifolds are actually $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{4,4,2}$ and $\mathbb{P}^1_{6,3,2}$. Another remark is that we can not obtain the LG/CY correspondence for all invertible simple elliptic singularities simply in current stage only because we can not compute all the FJRW invariants except the good cases, under the current technology.

Recently, the method has been generalized to solve the LG/CY correspondence for the maximal quotient of a Fermat quintic 3-fold [IMRS].

1.2 Two reconstruction theorems

As a first step for understanding formula (1.1), let us say more about the two generating functions in this formula. Using tautological relations on cohomology of moduli space of stable curves and the axioms of Gromov-Witten theory, we prove a reconstruction theorem for Gromov-Witten invariants of all elliptic $\mathbb{P}^1$-orbifolds in Chapter III. A similar theorem is carried out for the corresponding FJRW theory in Chapter IV.

Theorem I.3. *[KS] We have the following two reconstruction statements:*

(1) *The Gromov-Witten generating function of* $\mathcal{X} = \mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{4,4,2}, \mathbb{P}^1_{6,3,2}$ *is uniquely reconstructed from the following initial data: the Poincaré pairing, the Chen-Ruan product, and an initial correlator*

$$\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle^{\mathcal{X}}_{0,3,1} = 1.$$

Here $\Delta_{i,1}$ *are twisted sectors in the Chen-Ruan cohomology of* $\mathcal{X}$ *that support on the i-th orbifold point with a smallest degree shifting number.*

(2) *For an invertible simple elliptic singularity W the FJRW generating function of* (W, G_W) *is uniquely reconstructed from the pairing, the FJRW ring structure constants and some* 4*-point basic correlators with one of the insertions being a top degree element.*

The first reconstruction theorem is proved in [KS], the second one is proved in [KS] for three cubic polynomial cases and then generalized to all other cases in [MS]. For most of invertible simple elliptic singularities with maximal diagonal symmetry group, we can compute the FJRW ring structure and those basic 4-point correlators. However, as we already pointed out, there are three examples out of the reach of the current technology.

For Ruan's conjecture to make sense, we need the generating function to be analytic with respect to the Kähler parameter. This is often a difficult problem in Gromov-Witten theory and interesting in its own right. Our next theorem (Theorem I.4) establishes it for both Gromov-Witten theory and FJRW theory. It is convenient to consider the *ancestor correlator function* $\langle\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n) \rangle\rangle^{GW}_{g,n}(\mathbf{t})$. (See the precise definition in Chapter II.) Among all the cases we consider, the state space is decomposed into $H^{<2} \oplus H^2$ where H^2 is a one-dimensional space of degree 2 classes and $H^{<2}$ is the subspace of degree < 2. Let $\mathbf{t} = (s, t)$ with $s \in H^{<2}$ and $t \in H^2$. With out losing generality, we also view $\mathbf{t}$ as a complex valued vector once a basis of the state space is fixed. We can convert t to the familiar q variable by the substitution $q = e^t$. We define $\langle\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n) \rangle\rangle^{FJRW}_{g,n}(\mathbf{t})$ in the same way. The main difference is the absence of the β variable. It is obvious that for invertible simple elliptic singularity W with a group G_W, H^2 is always one-dimensional.

Theorem I.4. *[KS] We have the following two convergence statements:*

(1) For the above three classes of elliptic orbifold $\mathbb{P}^1$'s,

$$\langle\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n)\rangle\rangle_{g,n}^{GW}(s,t)$$

converges to an analytic function near $s = (0, \cdots, 0)$, $Re(t) \ll 0$ or $q = 0$.

(2) For its FJRW counterparts,

$$\langle\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n)\rangle\rangle_{g,n}^{FJRW}(s,t)$$

converges to an analytic function near $s = (0, \cdots, 0)$, $t = 0$.

(1) is often referred to as the convergence at the large volume limit $\mathbf{t} = (0, \cdots, 0, -\infty)$, while (2) can be referred to as the convergence at small volume limit $\mathbf{t} = (0, \cdots, 0)$.

1.3 Global mirror symmetry

Chiodo-Ruan [ChiR3] has reframed Ruan's conjecture in the language of *global mirror symmetry*. Let us explain their approach.

Global B-model

The global mirror symmetry of our examples involves global B-model objects. We consider Saito-Givental theory for a one-parameter family

$$W_\sigma = W + \sigma\phi_{-1}.$$

Here W is an invertible simple elliptic singularity and ϕ_{-1} is a top degree non-vanishing monomial in the Jacobi algebra $\mathscr{Q}_W$. Let us fix an invertible simple elliptic singularity W

of type $E^{(1,1)}_{\mu-2}$, $\mu = 8, 9, 10$. Saito constructed a flat structure on the miniversal deformation space $\mathcal{M}$ of W using primitive forms [Sa1]. The deformation along ϕ_{-1} is called *marginal deformation*. Following Givental's higher-genus reconstruction formalism [Gi2], we define for every semisimple point $\mathbf{s}$ in $\mathcal{M}$ a formal power series $\mathcal{A}_{\mathbf{s}}(\hbar; \mathbf{q})$ called the *total ancestor potential* of W. More details of the Saito-Givental theory will be introduced later. Let us point out that the primitive form depends on the choice of W and the choice of marginal element ϕ_{-1}. In [MR], Milanov and Ruan have worked out a *global* Saito-Givental theory in the sense of allowing the parameter σ to vary. Let $p_1, \ldots, p_l$ be the points on the complex line, s.t., for $\sigma = p_i$ the point $\mathbf{X} = (X_1, X_2, X_3) = (0, 0, 0)$ is not an isolated critical point of the polynomial $W_\sigma = W + \sigma\phi_{-1}$. The points

$$0,\ p_1, \ldots, p_l,\ \infty \in \mathbb{C} \cup \{\infty\}$$

will be called *special limit points* in our setting. Especially, $\sigma = 0$ is always called a *Gepner point*. We can classify all the special limit points into two different types, according to the local monodromy on a two dimensional subspace of the middle dimensional cohomology of the vanishing cycles of W. We say the special limit point is of *large complex structure limit* type if the local monodromy is maximal unipotent. We say the special limit point is of *Landau-Ginzburg type* is the local monodromy is diagonalizable. Our goal is to study the total ancestor potentials at the special limit points.

Berglund-Hübsch-Krawitz mirror construction

It turns out the Saito-Givental theory for an invertible simple elliptic singularity is related to the FJRW theory by a simple and elegant mirror construction, for Landau-Ginzburg model. Now we refer this construction as the Berglund-Hübsch-Krawitz mirror (or the BHK mirror for short). For an invertible polynomial W, its *transpose polynomial*

W^T is the unique invertible polynomial such that $E_{W^T} = (E_W)^T$. Thus for

$$W = \sum_{i=1}^{N} \prod_{j=1}^{N} X_i^{a_{ij}},$$

we have

$$W^T = \sum_{i=1}^{N} \prod_{j=1}^{N} X_i^{a_{ji}}.$$

The role of the transpose polynomial W^T in mirror symmetry was first studied by Berglund and Hübsch (see [BH]). Krawitz then introduced a mirror group construction G^T [Kr]. (W^T, G^T) is considered to be the BHK mirror for (W, G). In state space level, this refers to as the FJRW ring of (W, G) is isomorphic to the orbifold Jacobi algebra of (W^T, G^T). This is widely proved for various singularities in state space level with a Frobeniu algebra structure, see [FJR2, Kr, KP+, FS, Ac, KS, FJJS]. When $G = G_W$, then $G_W^T = \{1\}$ is the trivial group which contains only the identity element. And the data appears in B-model for $(W^T, \{1\})$ is the Saito-Givental theory.

LG-to-LG mirror theorem

We can study the special limits in Saito-Givental theory for one-parameter families of simple elliptic singularities, $W^T + \sigma\phi_{-1}$. It was conjectured that the Gepner point should always has a geometric mirror, its mirror FJRW theory for (W, G_W). We prove this holds true at least for those FJRW theories which are computable so far. This is the so called *LG-to-LG mirror theorem*. (This is a generalization of Witten's mirror conjecture for ADE-singularities to elliptic cases, see [Ru] and [ChiR3].)

Theorem I.5. *[KS, MS] Let W be a good invertible simple elliptic singualrities, we can choose the coordinates appropriately, such that*

$$\mathcal{A}_W^{FJRW} = \mathcal{A}^{SG}_{W^T_{\sigma=0}}. \tag{1.2}$$

This thoerem was proved when W is one of three cubic cases in [KS] and later generalized to all other cases in [MS].

LG-to-CY mirror theorem

Another mirror theorem focuses on those special points of large complex structure limits. They contain all other points with finite values and some $\sigma = \infty$ in some cases. We call it *LG-to-CY mirror theorem.*

Theorem I.6. *[KS, MR, MS] Let $\mathcal{X}$ be an elliptic orbifold $\mathbb{P}^1$ and its Chen-Ruan cohomology space has rank μ. There exists an invertible simple ellitpic singularity $W \in E^{(1,1)}_{\mu-2}$ and a special point σ of large complex structure limit type, such that we can choose a coordinate system and have*

$$\mathcal{A}^{GW}_{\mathcal{X}} = \mathcal{A}^{SG}_{W_\sigma}. \tag{1.3}$$

Again, this was first proved for three cubic type singularities and their special limits at $\sigma = \infty$ in [KS, MR]. It was also proved via Fermat type singularities at special limits of finite values in [MS].

Classification of special limits

As we discribed above, we have more special limits than those appeared in the previous theorems. For example, there are special limits at $\sigma = \infty$ with diagonalizable local monodromy. $W_\sigma = X_1^6 + X_2^3 + X_3^2 + \sigma X_1^4 X_2$ is such an example. Since the local monodromy is diagonalizable, it is indicated that it might be mirror to some FJRW theory. However, the BHK mirror is no longer the correct mirror in this example. On the Saito-Givental side, we can compute the total ancestor potential at all special limits. It is conjectured in [MS] that it is enough to extract information of the mirrors only from the j-invariant of the elliptic curve E_σ at the special value and μ, the Milnor number.

Conjecture I.7. *[MS] All special limits appear in Saito-Givental theories for invertible simple elliptic singularities are classified by the Milnor number μ of the singularity and j-invariant of the elliptic curve at the special limit.*

In particular, we have three different choices of Milnor numbers, $\mu = 8, 9$ or 10. This only depends on the choice of W. The values of the j-invariant at a special limit point can be only $0, 1728$ or ∞. This however depends on the choice of marginal direction and the value of σ. Overall, we have nine different types of special limits for all invertible simple elliptic singularities. For each Milnor number $\mu = 8, 9, 10$, there are one type of GW-limit and two different types of FJRW-limit. The special limit is a GW-limit if and only if the j-invariant is ∞. In this paper, we will give a proof of this conjecture for the Fermat type singularity. A complete proof of this conjecture will apear in a future porject.

1.4 Modularity

A remarkable phenomenon in Gromov-Witten theory is the appearance of (quasi) modular forms. A Gromov-Witten generating function can be thought as a counting function for the virtual number of holomorphic curves, i.e., one-dimensional objects. Therefore, it is natural to speculate if modular forms appear here too. Indeed, this strategy has been carried out for elliptic curves [Di, OP], some K-3 surfaces [BL] and the so called reduced Gromov-Witten theory of K3-surfaces [MPT]. In the middle of the 90's, by studying the physical B-model of Gromov-Witten theory, Bershadsky, Cecotti, Ooguri and Vafa boldly conjectured that the Gromov-Witten generating function of any Calabi-Yau manifolds are in fact *quasi-modular forms*. A key idea in [BCOV] is that the B-model Gromov-Witten function should be modular but non-holomorphic. Furthermore, its anti-holomorphic dependence is governed by the famous *holomorphic anomaly equations*. During the last decade, Klemm and his collaborators have put forth a series of papers to solve the holomorphic anomaly equations [ABK, HKQ]. Motivated by the physical intuition, there were two independent works recently in mathematics to establish the modularity of Gromov-Witten theory rigorously for local $\mathbb{P}^2$ [CI2] and elliptic orbifolds $\mathbb{P}^1$ [KS, MR]. In fact, the later result was generalized to a cycle-valued version of modularity in [MRS]. Let's briefly describe it.

For a projective variety X, we can construct *Gromov-Witten cycles (cohomological field theories)* $\Lambda^X_{g,n,\beta}(\gamma_1,\cdots,\gamma_n)$ by a partial integration, see formula (2.4) in Chapter II. Here $\overline{\mathcal{M}}_{g,n}$ is the Deligne-Mumford compactification of the moduli space of stable curves, see [DeM]. The degree of the cycle is computed from the *dimension axiom*,

$$\deg_{\mathbb{C}} \Lambda^X_{g,n,\beta}(\gamma_1,\cdots,\gamma_n) = (g-1)\dim_C(X) + \sum_{i=1}^n \deg_{\mathbb{C}}(\gamma_i) - c_1(TX)\cdot\beta.$$

The numerical Gromov-Witten invariants are obtained by

$$\langle \tau_{\iota_1}(\gamma_1),\cdots,\tau_{\iota_c}(\gamma_n)\rangle^X_{g,n,\beta} = \int_{\overline{\mathcal{M}}_{g,n}} \Lambda^X_{g,n,\beta}(\gamma_1,\cdots,\gamma_n)\cup\prod_{i=1}^n \psi_i^{\iota_i}.$$

Motivated by the corresponding work in number theory [Z], we want to consider the generating function of Gromov-Witten cycles

$$\left(\Lambda^X_{g,n}(q)\right)(\gamma_1,\cdots,\gamma_n) = \sum_{\beta\in \mathrm{NE}(X)} \Lambda^X_{g,n,\beta}(\gamma_1,\cdots,\gamma_n)\, q^\beta. \tag{1.4}$$

We view the RHS of (1.4) as a function on q taking value in $H^*(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$. To emphasise this perspective, we sometimes refer to it as *cycle-valued* generating function. The main theorem in [MRS] is

Theorem I.8. *[MRS] Suppose that $\mathcal{X}$ is one of the three elliptic orbifolds $\mathbb{P}^1$ with three non-trivial orbifold points; then $\left(\Lambda^X_{g,n}(q)\right)(\gamma_1,\cdots,\gamma_n)$ converges to a cycle-valued quasi-modular form of an appropriate weight for a finite index subgroup $\Gamma(N)$ of $SL_2(\mathbb{Z})$ under the change of variables $q = e^{2\pi i\tau/N}$, where $N = 3,4,6$ respectively.*

We should mention that the above cycle-valued modularity theorem is not yet known for elliptic curve.

We obtain the modularity of numerical Gromov-Witten invariants by integrating the $\Lambda^{\mathcal{X}}_{g,n}(\gamma_1, \cdots, \gamma_n)$ with psi-classes over the fundamental cycle $[\overline{\mathcal{M}}_{g,n}]$. On the other hand, we can also use other interesting classes of $\overline{\mathcal{M}}_{g,n}$ such as κ_i's or Hodge class λ_i's.

Suppose that P is a polynomial of $\psi_i, \kappa_i, \lambda_i$. We define a *generalized numerical Gromov-Witten invariants*

$$\langle \gamma_1, \cdots, \gamma_n; P\rangle^{\mathcal{X}}_{g,n,\beta} = \int_{\overline{\mathcal{M}}_{g,n}} P \cup \Lambda^{\mathcal{X}}_{g,n,\beta}(\gamma_1, \cdots, \gamma_n)$$

and its generating function

$$\langle \gamma_1, \cdots, \gamma_n; P\rangle^{\mathcal{X}}_{g,n}(q) = \sum_{\beta \in \mathrm{NE}(\mathcal{X})} \langle \gamma_1, \cdots, \gamma_n; P\rangle^{\mathcal{X}}_{g,n,\beta}\, q^{\beta}.$$

Here, we set it to be zero if the dimension constraint are not satisfied.

Corollary I.9. *Suppose that $\mathcal{X}$ is one of the above three elliptic orbifolds $\mathbb{P}^1$. Then, the above generalized numerical Gromov-Witten generating functions are quasi-modular forms for the same modular group and weights given by the main theorem.*

The proof of the numerical version consists of two steps, see [MR]. The first step is to construct a higher genus B-model theory (modulo an extension problem) and prove its modularity. Then, the second step is to prove mirror theorems to match it with a Gromov-Witten theory which will solve the extension property as well as inducing the modularity for a Gromov-Witten theory. The same strategy can be carried out on the cycle level. The main new ingredient is Teleman's reconstruction theorem [Te].

1.5 Detailed outline

In chapter 2, we first introduce the orbifold Gromov-Witten theory as a cohomological field theory. Then we decribe our main targets, three types of elliptic orbifold projective lines $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{4,4,2}$, $\mathbb{P}^1_{6,3,2}$ and compute their Chen-Ruan product. Then we obtain the key theorem of this chapter, that the ancestor Gromov-Witten invariants of all genera for those orbifolds are uniquely determined by their Chen-Ruan product and one extra single nonzero correlator of genus zero, three point and degree one. We end up chapter 2 by giving a convergence result for the Gromov-Witten ancestor correlator functions. In chapter 3, we give a parallel discussion on the Fan-Jarvis-Ruan-Witten theory for simple elliptic singularities. We introduce the axioms of FJRW theory for general hypersurface singularities, compute their FJRW rings. Moreover, we also give a classification for those ring structures. We end up this chapter by proving a reconstruction theorem for all of the FJRW invariants and introducing a convergence result for those FJRW correlator functions. In chapter 4, we introduce Saito's construction of Frobenius manifold on minversal deformations of invertible simple elliptic singularities. We describe Milanov-Ruan's result on its global theory. We compute the B-model initial correlators by choosing flat coordinates, based on analying the Picard-Fuchs equations for all sectors in the global B-model. We also discuss how to obtain the mirror for geometric A-model theory by extension of Saito-Givental theory to a non-semisimple point. In Chapter 5, we give the Berglund-Hübsch-Krawitz mirror construction and prove the LG-to-LG mirror symmetry theorem of all Gepner points in invertible simple elliptic singualrities. In Chapter 6, we discuss the global picture for mirror symmetry by analyzing other special points in B-model. We prove the LG-to-CY mirror theorem for Fermat types. In Chapter 7, we discuss Givental's quantization formula and give a proof for the LG/CY correspondence for elliptic orbifold $\mathbb{P}^1$ by analyzing the quanzation formula and analytic continuation on the global B-model. In Chapter 8, we introduce the modularity in global B-model, compute the modular group for three examples and prove the modularity theorem for Gromov-Witten theory of those elliptic orbifold $\mathbb{P}^1$ by using mirror symmetry. Most of the statements are in cycle-valued version. In Chapter 9, we give the proof of convergence theorem for both Gromov-Witten theory and Fan-Jarvis-Ruan-Witten theory of our examples. This complete the statement that our mirror theorems extend to the non-semisimple points we want. In the appendix, we give the recursion formula for basic correlators in Gromov-Witten theory of elliptic orbifold $\mathbb{P}^1$. This completes our reconstruction theorem in the Gromov-Witten theory.

CHAPTER II

Gromov-Witten theory for elliptic orbifolds $\mathbb{P}^1$

2.1 Cohomological Field Theories

We recall $\overline{\mathcal{M}}_{g,n}$ is the Deligne-Mumford compactification of the moduli space of genus g stable curves with n marked points, see [DeM]. Let H be a vector space of dimension N with a unit $\mathbf{1}$ and a non-degenerate paring $\eta : H \times H \to \mathbb{C}$. Without loss of generality, we always fix a basis of H, say

$$\mathscr{S} := \{\partial_0, \cdots, \partial_{N-1}\},$$

and we set $\partial_0 = \mathbf{1}$. Let $\left(\partial^j\right)$ be the dual basis in the dual space $H^\vee$. A *cohomological field theory* (or CohFT for short) is a set of multi-linear maps $\Lambda = \{\Lambda_{g,n}\}$, with

$$\Lambda_{g,n} : H^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C}),$$

or equivalently,

$$\Lambda_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C}) \otimes (H^\vee)^{\otimes n},$$

defined for any g, n such that $2g - 2 + n > 0$. Furthermore, Λ satisfies a set of axioms (CohFT axioms) described below:

1. (S_n-invariance) For any $\sigma \in S_n$, and $\gamma_1, \ldots, \gamma_n \in H$,

$$\Lambda_{g,n}(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)}) = \Lambda_{g,n}(\gamma_1, \ldots, \gamma_n).$$

2. (Gluing tree) Let

$$\rho_{tree} : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$$

where $g = g_1 + g_2, n = n_1 + n_2$, be the morphism induced from gluing the last marked point of the first curve and the first marked point of the second curve; then

$$\begin{aligned}&\rho_{tree}^*(\Lambda_{g,n}(\gamma_1, \ldots, \gamma_n))\\ &\quad = \sum_{\alpha,\beta\in\mathscr{S}} \Lambda_{g_1,n_1+1}(\gamma_1, \ldots, \gamma_{n_1}, \alpha)\eta^{\alpha,\beta}\Lambda_{g_2,n_2+1}(\beta, \gamma_{n_1+1}, \ldots, \gamma_n).\end{aligned}$$

Here $(\eta^{\alpha,\beta})_{N\times N}$ is the inverse matrix of $(\eta(\alpha,\beta))_{N\times N}$.

3. (Gluing loop) Let

$$\rho_{loop} : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n},$$

be the morphism induced from gluing the last two marked points; then

$$\rho_{loop}^*(\Lambda_{g,n}(\gamma_1, \ldots, \gamma_n)) = \sum_{\alpha,\beta\in\mathscr{S}} \Lambda_{g-1,n+2}(\gamma_1, \ldots, \gamma_n, \alpha, \beta)\eta^{\alpha,\beta}.$$

4. (Pairing)

$$\int_{\overline{\mathcal{M}}_{0,3}} \Lambda_{0,3}(\mathbf{1}, \gamma_1, \gamma_2) = \eta(\gamma_1, \gamma_2).$$

If in addition the following axiom holds:

(5) (Flat identity) Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful morphism; then

$$\Lambda_{g,n+1}(\gamma_1, \ldots, \gamma_n, \mathbf{1}) = \pi^*\Lambda_{g,n}(\gamma_1, \ldots, \gamma_n).$$

then we say that Λ is a CohFT with a *flat identity*.

If Λ is a CohFT; then there is a natural formal family of CohFTs. Namely,

$$\Lambda_{g,n}(\mathbf{t})(\gamma_1, \ldots, \gamma_n) = \sum_{l=0}^{\infty} \frac{1}{l!}\pi_* \Lambda_{g,n+l}(\gamma_1, \ldots, \gamma_n, \mathbf{t}, \ldots, \mathbf{t}), \quad \mathbf{t} \in H$$

where $\pi : \overline{\mathcal{M}}_{g,n+l} \to \overline{\mathcal{M}}_{g,n}$ is the morphism forgetting the last l marked points. Note that $\Lambda_{0,3}(\mathbf{t})$ will induce a family of Frobenius multiplications $\bullet_{\mathbf{t}}$ on (H, η), defined by

$$\eta(\alpha \bullet_{\mathbf{t}} \beta, \gamma) = \int_{\overline{\mathcal{M}}_{0,3}} \Lambda_{0,3}(\mathbf{t})(\alpha, \beta, \gamma). \tag{2.1}$$

The CohFT axioms imply that $(H, \eta, \bullet_{\mathbf{t}})$ is a Frobenius manifold in the sense of Dubrovin [Du]. The vector space H is called the *state space* of the CohFT.

Examples of CohFTs

Let $\mathbb{C}^N$ be the complex vector space equipped with the standard bi-linear pairing: $(e_i, e_j) = \delta_{i,j}$. Let $\Delta = (\Delta_1, \cdots, \Delta_N)$ be a sequence of *non-zero* complex numbers. The following definition

$$I^{N,\Delta}_{g,n}(e_{i_1}, \ldots, e_{i_n}) := \begin{cases} \Delta_i^{g-1+\frac{n}{2}} \mathrm{PD}[\overline{\mathcal{M}}_{g,n}] \in H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{C}) & \text{if } i = i_1 = i_2 = \cdots = i_n, \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

induces a CohFT on $\mathbb{C}^N$ which we call a rank N *trivial* CohFT. Here $[\overline{\mathcal{M}}_{g,n}]$ is the fundamental cycle of $\overline{\mathcal{M}}_{g,n}$ and PD represents the Poincaré dual. The Frobenius algebra underlying $I^{N,\Delta}$ will be denoted by $(\mathbb{C}^N, \Delta)$. Using the Kronecker symbol δ_{ij}, we note that the Frobenius multiplication is given by

$$e_i \bullet e_j = \delta_{ij} \sqrt{\Delta_i}\, e_i.$$

The total ancestor potential of a CohFT

For a given CohFT Λ, the *ancestor correlator functions* are, by definition, the following formal power series in $\mathbf{t} \in H$:

$$\langle\langle \tau_{k_1}(\alpha_1), \ldots, \tau_{k_n}(\alpha_n) \rangle\rangle_{g,n}(\mathbf{t}) = \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{g,n}(\mathbf{t})(\alpha_1, \ldots, \alpha_n)\, \psi_1^{k_1} \ldots \psi_n^{k_n}, \tag{2.3}$$

where $\alpha_i \in H$, $k_i \in \mathbb{Z}_{\geq 0}$ and ψ_i is the i-th ψ-class on $\overline{\mathcal{M}}_{g,n}$. The value of a correlator function at $\mathbf{t} = 0$ is called simply a *correlator* and we denote by $\langle \tau_{k_1}(\alpha_1), \ldots, \tau_{k_n}(\alpha_n) \rangle_{g,n}$ only. We call g the genus of the correlator function and each $\tau_{k_i}(\alpha_i)$ is called a *descendant* (resp. *non-descendant*) *insertion* if $k_i > 0$ (resp. $k_i = 0$).

For each basis vector ∂_i in H, we fix a sequence of formal variables $\{q^i_k\}_{k=0}^{\infty}$ and define

$$\mathbf{q}(z) = \sum_{k=0}^{\infty} \sum_{i=0}^{N-1} q^i_k \, \partial_i \, z^k \ ;$$

then the *genus-g ancestor potential* is the following generating function:

$$\mathcal{F}^{\mathcal{X}}_g(\mathbf{q}, \mathbf{t}) := \sum_n \frac{1}{n!} \langle\langle \mathbf{q}(\psi_1) + \psi_1, \ldots, \mathbf{q}(\psi_n) + \psi_n \rangle\rangle^{\mathcal{X}}_{g,n}(\mathbf{t}),$$

where each correlator should be expanded multi-linearly in $\mathbf{q}$ and the resulting correlators are evaluated according to (2.3). Let us point out that we have assumed that the CohFT has a flat identity $\mathbf{1} \in H$ and we have incorporated the *dilaton shift* in our function, so that $\mathcal{F}^{GW}_g$ is a formal series in q_k, $k \neq 0$ and $q_1 + \mathbf{1}$. Finally, the *total ancestor potential* is

$$\mathcal{A}^{\mathcal{X}}(\hbar; \mathbf{q}, \mathbf{t}) := \exp\left(\sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}^{\mathcal{X}}_g(\mathbf{q}, \mathbf{t}) \right).$$

.

2.2 Orbifold Gromov-Witten theory

For simplicity, we assume $\mathcal{X}$ is a compact Kahlër orbifold, which is a quotient space of a Kahlër manifold Y by a faithful finite abelian group action, i.e. $\mathcal{X} = Y/G$.

The *inertia orbifold* of $\mathcal{X}$, $I\mathcal{X}$, which is defined by

$$I\mathcal{X} := \coprod_{(g) \in G^*} \mathrm{Fix}(g)/G.$$

Here g is an element in the finite group G, and G^* is the set of conjugacy classes of G. Let us use (g) to represent the conjugacy class of g. Fix(g) is the set of fixed points in Y under the action of g. Since G is abelian, G is also the centralizer of g.

The *Chen-Ruan cohomology* is defined by

$$H^*_{\mathrm{CR}}(\mathcal{X}) = \bigoplus_{(g) \in G^*} H^{\tilde{*}}(\mathrm{Fix}(g)) \, .$$

$H^{*}(\mathrm{Fix}(g))$ is the ordinary De Rham cohomology of $\mathrm{Fix}(g)$, with a twisting on the degree, defined by the group action on Y. Let us recall the action of g on the tangent space TY of Y. Since G is a finite abelian group and the action of g is faithful, the action of g on the target space can be viewed as a scalar multiplication on $\mathbb{C}^N$, where N is the complex dimension of Y. We denote the action by a N-tuple of nonzero complex numbers

$$\left(\exp(2\pi\sqrt{-1}\Theta_1^g), \exp(2\pi\sqrt{-1}\Theta_2^g), \cdots, \exp(2\pi\sqrt{-1}\Theta_N^g)\right) \in \mathbb{C}^N, \quad \Theta_i^g \in \mathbb{Q}/\mathbb{Z}, 1 \leq i \leq N.$$

So for a class of the Chen-Ruan cohomology in the component $\mathrm{Fix}(g)$, its complex degree is the complex degree of the class as a De Rham cohomology element plus the *degree shifting number*

$$\iota_g := \sum_{i=1}^{N} \Theta_i^g.$$

If the element g acts trivially, we see that the degree shifting number is zero. Otherwise, it is not zero and all the classes with nonzero degree shifting number are called *twisted sectors*.

We define $\overline{\mathcal{M}}^{\mathcal{X}}_{g,n,\beta}$ to be the moduli space of all stable maps $f : C \longrightarrow \mathcal{X}$, from a genus-$g$ orbi-curve C, equipped with n marked points, to $\mathcal{X}$, such that $f_*([C]) = \beta \in \mathrm{NE}(\mathcal{X})$. Here $[C]$ is the fundamental class of curve C. Let us denote by π the forgetful map, and by ev_i the evaluation at the i-th marked point

$$\overline{\mathcal{M}}_{g,n} \xleftarrow{\pi} \overline{\mathcal{M}}^{\mathcal{X}}_{g,n+k,\beta} \xrightarrow{\mathrm{ev_i}} I\mathcal{X}\ .$$

The moduli space is equipped with a virtual fundamental cycle $\left[\overline{\mathcal{M}}^{\mathcal{X}}_{g,n,\beta}\right]^{\mathrm{vir}} \in H_*\left(\overline{\mathcal{M}}^{\mathcal{X}}_{g,n,\beta}\right)$, such that

$$\Lambda^{\mathcal{X}}_{g,n,\beta} : H^*_{\mathrm{CR}}(\mathcal{X})^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{C})$$

defined by

$$\Lambda^{\mathcal{X}}_{g,n,\beta}(\alpha_1, \ldots, \alpha_n) := \pi_*\left(\left[\overline{\mathcal{M}}^{\mathcal{X}}_{g,n,\beta}\right]^{\mathrm{vir}} \cap \prod_{i=1}^{n} \mathrm{ev}_i^*(\alpha_i)\right). \tag{2.4}$$

We put those classes together with Novikov variable q^β and define an element

$$\Lambda^{X}_{g,n}(q) = \sum_{\beta} \Lambda^{X}_{g,n,\beta}\, q^\beta : H^*_{\mathrm{CR}}(\mathcal{X})^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{C}[q^{NE(\mathcal{X})}]) \tag{2.5}$$

The Novikov variables satisfies the following rule, for $\beta_1, \beta_2 \in NE(\mathcal{X})$,

$$q^{\beta_1} q^{\beta_2} = q^{\beta_1+\beta_2}.$$

$\Lambda_{g,n}(q)$ forms a CohFT with state space $H^*_{\mathrm{CR}}(\mathcal{X})$. The total ancestor potential $\mathcal{A}^{\mathcal{X}}$ of $\mathcal{X}$ is by definition the total ancestor potential of the CohFT.

From now on, we assume $H^{1,1}(\mathcal{X}, \mathbb{C}) = \mathbb{C}$, i.e. the Kähler moduli is one dimensional. This simplifies our notation for Novikov variables. As any $\beta \in H_2(\mathcal{X})$ can be view as

$$\beta = d \cdot \mathrm{PD}(\mathcal{P})$$

for unique nonnegative integer d. Here $\mathrm{PD}(\mathcal{P})$ is the Poincaré dual of a generator $\mathcal{P} \in H^{1,1}(\mathcal{X}, \mathbb{Z})$. For simplicity, we use the index d to represent β. We set $\Lambda^{X}_{g,n,d} = \Lambda^{X}_{g,n,\beta}$ and $q^d = q^\beta$. Now we get an example of a family of CohFTs coming from Gromov-Witten theory, parametrized by a variable $q \in \mathbb{C}$.

$$\Lambda^{X}_{g,n}(q) = \sum_{d \geq 0} \Lambda^{X}_{g,n,d}\, q^d \tag{2.6}$$

Let H be the Chen-Ruan cohomology $H^*_{\mathrm{CR}}(\mathcal{X})$, η be the Poincaré pairing. There exist a well defined CohFT $\Lambda^{X}_{g,n}(q)$ at $q = 0$. The above axioms make sense for cohomology classes $\Lambda^{X}_{g,n}(q)$ that have coefficients in some ring of formal power series. In such a case we say that we have a *formal* cohomological field theory. A priori, the CohFT $\Lambda^{X}_{g,n}(q)$ is only formal.

There is a quantum product structure $\star_q$ on Chen-Ruan cohomology on $H^*_{\mathrm{CR}}(\mathcal{X})$. It was called the *quantum Chen-Ruan product*, and defined by

$$\langle \alpha_1 \star_q \alpha_2, \alpha_3 \rangle = \sum_{\beta} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,\beta}\, q^\beta$$

By restricting to $q = 0$, we define the *Chen-Ruan orbifold cup product* (or *Chen-Ruan product* for short), and we denote it by $\star$. By definition, the structure constants of $\star$ are defined by the following genus-0 degree-0 correlators:

$$\langle \alpha_1 \star \alpha_2, \alpha_3 \rangle = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,0}.$$

For more details on orbifold Gromov–Witten theory we refer to [ALR, CheR1, CheR2, AbGV]. We list some of the axioms for furture use.

- Dimension axiom. The virtual dimension of $\overline{\mathcal{M}}^{\mathcal{X}}_{g,n,\beta}$ is

$$\operatorname{vir}\dim_{\mathbb{C}} \overline{\mathcal{M}}^{\mathcal{X}}_{g,n,\beta} = (3-1)(g-1) + n + c_1(T\mathcal{X}) \cdot \beta = 2g - 2 + n. \tag{2.7}$$

- Divisor equation:

$$\langle \tau_{l_1}(\gamma_1) \cdots \tau_{l_n}(\gamma_n), \mathcal{P} \rangle_{g,n+1,d} = d \, \langle \tau_{l_1}(\gamma_1) \cdots \tau_{l_n}(\gamma_n) \rangle_{g,n,d}. \tag{2.8}$$

- String equation:

$$\langle \tau_{l_1}(\gamma_1) \cdots \tau_{l_n}(\gamma_n), \mathbf{1} \rangle_{g,n+1,d} = \sum_{i=1}^{n} \langle \tau_{l_1}(\gamma_1) \cdots \tau_{l_i-1}(\gamma_i) \cdots \tau_{l_n}(\gamma_n) \rangle_{g,n,d}. \tag{2.9}$$

- WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equation:

$$\frac{\partial^3 \mathscr{F}_0^{\mathcal{X}}}{\partial_1 \partial_2 \partial_i} \eta^{i,j} \frac{\partial^3 \mathscr{F}_0^{\mathcal{X}}}{\partial_j \partial_3 \partial_4} = \frac{\partial^3 \mathscr{F}_0^{\mathcal{X}}}{\partial_1 \partial_3 \partial_i} \eta^{i,j} \frac{\partial^3 \mathscr{F}_0^{\mathcal{X}}}{\partial_j \partial_2 \partial_4}. \tag{2.10}$$

2.3 Elliptic orbifolds $\mathbb{P}^1$

Three orbifolds points

Let $\mathbb{P}^1_{o_1,o_2,o_3}$ be the orbifold $\mathbb{P}^1$ with three orbifold points, such that, the i-th orbifold point has its isotropy group $\mathbb{Z}/o_i\mathbb{Z}$. In this paper, we are interested in the following 3 cases: $(o_1, o_2, o_3) = (3,3,3), (4,4,2), (6,3,2)$. Together with $\mathbb{P}^1_{2,2,2,2}$, they correspond to orbifold-$\mathbb{P}^1$s that are quotients of an elliptic curve by a finite group.

The Chen-Ruan cohomology $H^*_{\mathrm{CR}}(\mathbb{P}^1_{o_1,o_2,o_3})$ has the following form:

$$H^*_{\mathrm{CR}}(\mathbb{P}^1_{o_1,o_2,o_3}) = \left(\bigoplus_{i=1}^{3}\bigoplus_{j=1}^{o_i-1}\mathbb{C}[\Delta_{ij}]\right)\bigoplus\mathbb{C}[\Delta_{01}]\bigoplus\mathbb{C}[\Delta_{02}]. \tag{2.11}$$

where $\Delta_{01} = \mathbf{1}$ is the unit and $\Delta_{02} = \mathcal{P}$. The classes Δ_{ij} with $1 \le i \le 3, 1 \le j \le o_i - 1$ are in one-to-one correspondence with the twisted sectors, which come from orbifold points, and we define Δ_{ij} to be the unit in the cohomology of the corresponding twisted sector. In our context, the complex degrees are

$$\deg \Delta_{ij} = \frac{j}{o_i}, \quad 1 \le i \le 3, \quad 1 \le j \le o_i - 1.$$

The orbifold Poincaré pairing takes the form

$$\langle \Delta_{i_1 j_1}, \Delta_{i_2 j_2}\rangle = \begin{cases} \left(\delta_{i_1,i_2}\delta_{j_1+j_2,o_k}\right)/o_k, & k = i_1, i_1 + i_2 \neq 0; \\ \delta_{j_1+j_2,3}, & i_1 = i_2 = 0. \end{cases}$$

It is not hard to prove (using only the grading and the Poincaré pairing) that the above 3-point correlators are given by the following formulas:

$$\langle \Delta_{i_1 j_1}, \Delta_{i_2 j_2}, \Delta_{i_3 j_3}\rangle_{0,3,0} = \begin{cases} 1/o_i, & i_1 = i_2 = i_3 = k \in \{1,2,3\}, j_1 + j_2 + j_3 = o_k; \\ \langle \Delta_{i_2 j_2}, \Delta_{i_3 j_3}\rangle, & (i_1, j_1) = (0,1); \\ 0, & \text{otherwise}. \end{cases}$$

A degree 1 correlator

Lemma II.1. *For all $\mathcal{X} = \mathbb{P}^1_{3,3,3}, \mathbb{P}^1_{4,4,2}, \mathbb{P}^1_{6,3,2}$, we have*

$$\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1}\rangle^{\mathcal{X}}_{0,3,1} = 1. \tag{2.12}$$

Proof. For $\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1}\rangle^{\mathbb{P}^1_{3,3,3}}_{0,3,1}$, as in [ALR], we consider the R-equivalence class of principal $\mathbb{Z}/3\mathbb{Z}$-bundles over orbifold $\mathbb{P}^1_{3,3,3}$ with a $\mathbb{Z}/3\mathbb{Z}$-equivariant map to a genus-1 curve.

There is just one such equivalence class, thus $\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle_{0,3,1}^{\mathbb{P}^1_{3,3,3}} = 1$. The other two cases are obtained similarly. □

2.4 Reconstruction

We use WDVV, string equation, divisor equation (which does not exist in the FJRW theory) and other axioms in Gromov-Witten theory, to reconstruct higher genus (descendent) correlators from genus-0 primary correlators, and to reconstruct genus-0 primary correlators from genus-0 n-point basic correlators with degree at most 1, with $n \leq 3$. We will do the same thing in FJRW theory in next chapter. This technique is already used in [KS] for three special examples of simple elliptic singularities. As the reconstruction procedures used there only require tautological relations on cohomology of moduli spaces of curves, we can easily generalize to all other examples. We sketch the general procedures here. There are three steps.

First, we express the correlators of genus at least 2 and the correlators with descendant insertions in terms of correlators of genus-0 or genus-1 with non-descendant insertions (called *primary correlators*). This step is based on a tautological relation which splits a polynomial of ψ-classes and κ-classes with higher degree to a linear combination of products of boundary classes and polynomials of ψ-classes and κ-classes of lower degrees. This is called g-reduction. The reason why g-reduction works in our case is that the dimension axiom imposes a constraint on the degree of the polynomials involving ψ-classes and κ-classes (see Lemma II.5). In general, for an arbitrary CohFT this argument fails and one has to use other methods (e.g. Teleman's reconstruction theorem).

Next, we reconstruct the non-vanishing genus-1 primary correlators from genus 0 primary correlators using Getzler's relation. The latter is a relation in $H^4(\overline{\mathcal{M}}_{1,4})$, which gives identities involving the Gromov-Witten corrletors with genus 0 and 1. In order to obtain the desired reconstruction identity, i.e., to express genus-1 in terms of genus-0 correlators, one has to make an appropriate choice of the insertions corresponding to the 4 marked points in $\overline{\mathcal{M}}_{1,4}$.

Finally, we introduce the following definition

Definition II.2.

- We call a class γ *primitive* if it cannot be written as $\gamma = \gamma_1 \star \gamma_2$ for $0 < \deg \gamma_1 < \deg \gamma$.
- We call a correlator *basic* if there are no insertions of $\mathbf{1}, \mathcal{P}$ and at most two non-primitive insertions.
- We call a genus-0 primary correlator *reconstructable* if it can be expressed by linear combinations of products of $\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle_{0,3,1}^{\mathcal{X}}$ and Chen-Ruan product structural constants, only using WDVV, string and divisor equation.

To reconstruct the genus-0 correlators we use the WDVV equations. We use the WDVV equation to rewrite a primary genus-0 correlator which contains several non-primitive insertions to correlators with fewer non-primitive insertions and correlators with a fewer number of marked points. Again the dimension axiom should be taken into account in order to obtain a bound for the number of marked points. It turns out that all correlators are determined by the basic correlators with at most four marked points.

Main result

According to [KS],we have the following reconstruction result.

Lemma II.3. *The Gromov-Witten ancestor potential of* $\mathcal{X} = \mathbb{P}^1_{3,3,3}, \mathbb{P}^1_{4,4,2}, \mathbb{P}^1_{6,3,2}$ *is determined by the following initial data: the Poincaré pairing, the Chen-Ruan product, and the correlator* $\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle_{0,3,1}^{\mathcal{X}} = 1$.

$g \geq 2$

The key point for the higer genus reconstruction is the *g-reduction*. As we need the explicit form in the next subsection, we reproduce here. The *g-reduction* lemma is

Lemma II.4. *Let $P(\psi, \kappa)$ be a monomial in the ψ and κ-classes in $\overline{\mathcal{M}}_{g,n}$ of degree at least g for $g \geq 1$ or at least 1 for $g = 0$. Then the class $P(\psi, \kappa)$ can be represented by a linear combination of dual graphs, each of which has at least one edge.*

It was first used in [FSZ] for proving Witten's conjecture for r-spin curves. Then in [FJR2], it was generalized to case of central charge $\hat{c}_W \leq 1$ in the setting of FJRW theory, which includes the r-spin case as type A_{r-1} singularities $W := X^r$.

Now we apply this lemma to Gromov-Witten theory, we obtain

Lemma II.5. *For elliptic orbifold $\mathbb{P}^1$, the ancestor potential function is uniquely determined by the genus-0 potential and the genus-1 primary potential.*

Proof. We consider the Gromov-Witten invariants for the elliptic orbifold $\mathbb{P}^1$,

$$\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n), T_{i_1}, \cdots, T_{i_k} \rangle_{g,n+k,d} = \int_{\overline{\mathcal{M}}_{g,n+k}} \Psi_{l_1,\cdots,l_n} \cdot \Lambda^{\mathcal{X}}_{g,n+k,d}(\alpha_1, \cdots, \alpha_n, T_{i_1}, \cdots, T_{i_k}),$$

where $\Psi_{l_1,\cdots,l_n} = \prod_i \psi_i^{l_i}$. The correlator will vanish except for

$$\deg \Psi_{l_1,\cdots,l_n} + \sum_{i=1}^{n} \deg \alpha_i + \sum_{j=1}^{k} \deg(T_{i_j}) = 2g - 2 + n + k. \tag{2.13}$$

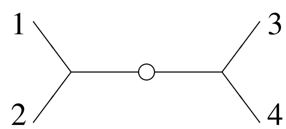
As long as $\deg \alpha_i \leq 1$ and $\deg T_{i_j} \leq 1$, we have $\deg \Psi_{l_1,\cdots,l_n} \geq 2g - 2$. Now we apply Lemma II.4. If $\deg \Psi_{l_1,\cdots,l_n}$ is large, then the integral is changed to the integral over the boundary classes while decreasing the degree of the total ψ-classes or κ-classes. After applying the splitting and composition laws, the genus involved will also decrease. We can continue this process until the original integral is represented by a linear combination of primary correlators of genus-0 and genus-1.

Moreover, for primary genus-1 correlators, we have $\deg \Psi_{l_1,\cdots,l_n} = 0$. Thus equation (2.13) holds if and only if $\deg \alpha_i = \deg T_{i_j} = 1$, i.e, we only need to consider genus-1 correlators of type $\langle \mathcal{P}, \cdots, \mathcal{P} \rangle_{1,n,d}$. □

$g = 1$

Getzler's relation

Here we prove the reconstruction theorem for primary genus-1 Gromov-Witten invariants for elliptic orbifold $\mathbb{P}^1$. Our main tool is the Getzler's relation. In [Ge], Getzler introduced a linear relation between codimension two cycles in $H_*(\overline{\mathcal{M}}_{1,4}, \mathbb{Q})$. Here we briefly introduce this relation for our purpose. Consider the dual graph,

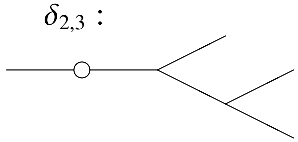
$\Delta_{12,34} =$ 

This graph represents a codimension-two stratum in $\overline{\mathcal{M}}_{1,4}$: A circle represents a genus-1 component, other vertices represent genus-0 components. An edge connecting two vertices represents a node, a tail (or half-edge) represents a marked point on the component of the corresponding vertex. $\Delta_{2,2}$ is defined to be the S_4-invariant of the codimension-two stratum in $\overline{\mathcal{M}}_{1,4}$,

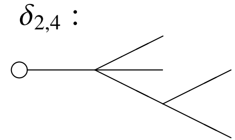
$$\Delta_{2,2} = \Delta_{12,34} + \Delta_{13,24} + \Delta_{14,23}.$$

We denote $\delta_{2,2} = [\Delta_{2,2}]$ the corresponding cycle in $H_4(\overline{\mathcal{M}}_{1,4}, \mathbb{Q})$. Other strata are defined similarly. For more details, see [Ge]. Here we list the corresponding unordered dual graph for each stratum,

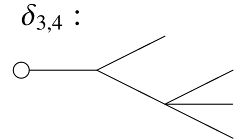
$\delta_{2,3}$:



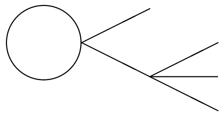
$\delta_{2,4}$:



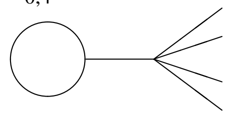
$\delta_{3,4}$:



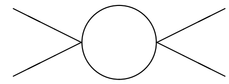
$\delta_{0,3}$:



$\delta_{0,4}$:



δ_{β} :



According to [Ge], Getzler's relation is as follows:

$$12\delta_{2,2} + 4\delta_{2,3} - 2\delta_{2,4} + 6\delta_{3,4} + \delta_{0,3} + \delta_{0,4} - 2\delta_{\beta} = 0. \tag{2.14}$$

For genus-1 correlators, one has the following:

Lemma II.6. *For $\mathcal{X} = \mathbb{P}^1_{3,3,3}, \mathbb{P}^1_{4,4,2}, \mathbb{P}^1_{6,3,2}$, the Getzler relation and divisor axiom imply that the genus-1 Gromov-Witten correlators of $\mathcal{X}$ can be reconstructed from genus-0 Gromov-Witten correlators.*

We consider the nonzero genus-1 correlator $\langle\gamma_1, \cdots, \gamma_n\rangle^{\mathcal{X}}_{1,n,d}$. As $\mathcal{X}$ is an elliptic orbifold $\mathbb{P}^1$ here, we have $\deg\gamma_i \leq 1$. According to the dimension formula (2.7), the correlator is nonzero only if every γ_i is $\mathcal{P}$. For $d > 0$, the genus-1 primary correlators are nonzero only if they are of type $\langle\mathcal{P}, \cdots, \mathcal{P}\rangle^{\mathcal{X}}_{1,n,d}$. By the divisor axiom, we have:

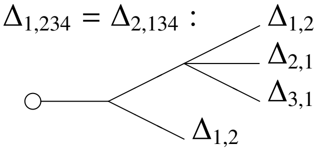
$$\langle\mathcal{P}, \cdots, \mathcal{P}\rangle^{\mathcal{X}}_{1,n,d} = d^{n-1}\langle\mathcal{P}\rangle^{\mathcal{X}}_{1,1,d}.$$

Remark II.7. $\langle\mathcal{P}, \cdots, \mathcal{P}\rangle^{\mathcal{X}}_{1,n,0} = 0$ for $n > 1$.

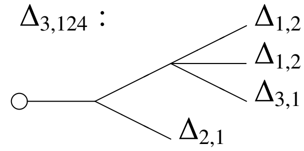
Now, we give the proof of Lemma II.6 by reconstructing $\langle\mathcal{P}\rangle^{\mathcal{X}}_{1,1,d}$, for any $d \geq 0$.

Proof. $\mathbb{P}^1_{3,3,3}$**-case:** We choose four insertions $\Delta_{1,2}, \Delta_{1,2}, \Delta_{2,1}, \Delta_{3,1} \in H^*_{\mathrm{CR}}(\mathbb{P}^1_{3,3,3})$, and we simply denote by $\Delta_{2,2;1;1}$. We integrate the class $\Lambda^{\mathbb{P}^1_{3,3,3}}_{1,4,d}(\Delta_{2,2;1;1})$ over codimension 2 strata of $\overline{\mathcal{M}}_{1,4}$. For $\delta_{3,4}$, the contribution comes from four decorated dual graphs:

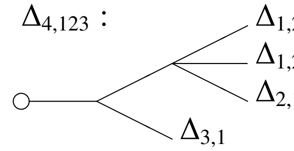
$\Delta_{1,234} = \Delta_{2,134}$:



$\Delta_{3,124}$:



$\Delta_{4,123}$:



Let us fix the total degree is $d+1$. Then

$$\begin{aligned}
&\int_{[\Delta_{1,234}]} \Lambda^{\mathbb{P}^1_{3,3,3}}_{1,4,d+1}(\Delta_{2,2;1;1}) \\
&= \sum_{d_1+d_2+d_3=d+1} \langle \mathcal{P} \rangle^{\mathbb{P}^1_{3,3,3}}_{1,1,d_1} \eta^{\mathcal{P},1} \langle 1, \Delta_{1,2}, \Delta_{1,1} \rangle^{\mathbb{P}^1_{3,3,3}}_{0,3,d_2} \eta^{\Delta_{1,1},\Delta_{1,2}} \langle \Delta_{2,2;1;1} \rangle^{\mathbb{P}^1_{3,3,3}}_{0,4,d_3} \\
&= \sum_{i=0}^{d+1} \langle \mathcal{P} \rangle^{\mathbb{P}^1_{3,3,3}}_{1,1,i} \langle \Delta_{2,2;1;1} \rangle^{\mathbb{P}^1_{3,3,3}}_{0,4,d+1-i}.
\end{aligned}$$

Then we use the result from genus-0 recursion that

$$\langle \Delta_{2,2;1;1} \rangle^{\mathbb{P}^1_{3,3,3}}_{0,4,0} = 0, \langle \Delta_{2,2;1;1} \rangle^{\mathbb{P}^1_{3,3,3}}_{0,4,1} = \frac{1}{3}.$$

Overall, we have

$$\int_{\delta_{3,4}} \Lambda^{\mathbb{P}^1_{3,3,3}}_{1,4,d+1}(\Delta_{2,2;1;1}) = \frac{4}{3} \langle \mathcal{P} \rangle^{\mathbb{P}^1_{3,3,3}}_{1,1,d} + 4 \sum_{i=0}^{d-1} \langle \mathcal{P} \rangle^{\mathbb{P}^1_{3,3,3}}_{1,1,i} \langle \Delta_{2,2;1;1} \rangle^{\mathbb{P}^1_{3,3,3}}_{0,4,d+1-i}. \tag{2.15}$$

Considering other strata in Getzler's Relation, the integration over $\delta_{2,2}, \delta_{2,3}$ and $\delta_{2,4}$ will all vanish for the following reasons:

- For $\delta_{2,2}$, $\langle \alpha, \beta, 1 \rangle^{\mathbb{P}^1_{3,3,3}}_{0,3,j} = 0$ for all $\{\alpha, \beta\} \subset \Delta_{2,2;1;1}$.
- For $\delta_{2,3}$, by dimension reason (2.7), $\langle \alpha \rangle^{\mathbb{P}^1_{3,3,3}}_{1,1,j} = 0$ for all $\alpha \in \Delta_{2,2;1;1}$.
- For $\delta_{2,4}$, by string equation, $\langle 1, \alpha, \beta, - \rangle^{\mathbb{P}^1_{3,3,3}}_{0,4,j} = 0$ for all $\{\alpha, \beta\} \subset \Delta_{2,2;1;1}$.

As the integration of $\Lambda^{\mathbb{P}^1_{3,3,3}}_{1,4,d+1}(\Delta_{2,2;1;1})$ over $\delta_{0,3}, \delta_{0,4}, \delta_\beta$ only give genus-0 invariants, the Getzler's relation implies $\langle \mathcal{P} \rangle^{\mathbb{P}^1_{3,3,3}}_{1,1,d}$ can be reconstructed from $\langle \mathcal{P} \rangle^{\mathbb{P}^1_{3,3,3}}_{1,1,d'}$ with $d' < d$ and genus-0 primary correlators.

$\mathbb{P}^1_{4,4,2}$-case: Now we choose four insertions $\Delta_{1,3}, \Delta_{1,2}, \Delta_{2,1}, \Delta_{3,1} \in H^*_{\mathrm{CR}}(\mathbb{P}^1_{4,4,2})$ and denote by $\Delta_{3,2;1;1}$. In this case, we use genus-0 computation:

$$\langle \Delta_{3,2;1;1} \rangle^{\mathbb{P}^1_{4,4,2}}_{0,4,0} = 0, \langle \Delta_{3,2;1;1} \rangle^{\mathbb{P}^1_{4,4,2}}_{0,4,1} = \frac{1}{4}.$$

Integrating $\Lambda^{\mathbb{P}^1_{4,4,2}}_{1,4,d+1}(\Delta_{3,2;1;1})$ on the $\delta_{3,4}$, we have

$$\int_{\delta_{3,4}} \Lambda^{\mathbb{P}^1_{4,4,2}}_{1,4,d+1}(\Delta_{3,2;1;1}) = \langle \mathcal{P} \rangle^{\mathbb{P}^1_{4,4,2}}_{1,1,d} + \sum_{i=0}^{d-1} \langle \mathcal{P} \rangle^{\mathbb{P}^1_{4,4,2}}_{1,1,i} \langle \Delta_{3,2;1;1} \rangle^{\mathbb{P}^1_{4,4,2}}_{0,4,d+1-i}.$$

Again, integrations over $\delta_{2,2},\delta_{2,3},\delta_{2,4}$ are zero and over $\delta_{0,3},\delta_{0,4},\delta_\beta$ only give genus-0 contribution. Thus Getzler's Relation implies $\langle\mathcal{P}\rangle_{1,1,d}^{\mathbb{P}^1_{4,4,2}}$ is reconstructable.

$\mathbb{P}^1_{6,3,2}$**-case:** Now we choose four insertions $\Delta_{1,5},\Delta_{1,2},\Delta_{2,1},\Delta_{3,1}\in H^*_{\mathrm{CR}}(\mathbb{P}^1_{6,3,2})$ and denote them as $\Delta_{5,2;1;1}$. In this case, we use genus-0 computation:

$$\langle\Delta_{5,2;1;1}\rangle_{0,4,0}^{\mathbb{P}^1_{6,3,2}}=0,\langle\Delta_{5,2;1;1}\rangle_{0,4,1}^{\mathbb{P}^1_{6,3,2}}=\frac{1}{6}.$$

Now we integrate $\Lambda_{1,4,d+1}^{\mathbb{P}^1_{6,3,2}}(\Delta_{5,2;1;1})$ over $\delta_{3,4}$,

$$\int_{\delta_{3,4}}\Lambda_{1,4,d+1}^{\mathbb{P}^1_{6,3,2}}(\Delta_{5,2;1;1})=\frac{2}{3}\langle\mathcal{P}\rangle_{1,1,d}^{\mathbb{P}^1_{6,3,2}}+4\sum_{i=0}^{d-1}\langle\mathcal{P}\rangle_{1,1,i}^{\mathbb{P}^1_{6,3,2}}\langle\Delta_{5,2;1;1}\rangle_{0,4,d+1-i}^{\mathbb{P}^1_{6,3,2}}.$$

Other strata only give genus-0 correlators. Thus $\langle\mathcal{P}\rangle_{1,1,d}^{\mathbb{P}^1_{6,3,2}}$ is reconstructable. □

$g=0$

To prove the genus 0 part, we first recall the WDVV equation for elliptic orbifold $\mathbb{P}^1$. Set $S=\{1,\cdots,n\}$, $n\geq 1$, for $d\geq 1$, we have:

$$\langle\gamma_1,\gamma_2,\delta_S,\gamma_3\star\gamma_4\rangle_{0,n+3,d}=I_0(n)+I_1(n)+I_2(n)+I_3(n) \tag{2.16}$$

where $|A|$ is the number of elements in the set A and

$$I_0(n)=\langle\gamma_1,\gamma_3,\delta_S,\gamma_2\star\gamma_4\rangle_{0,n+3,d}+\langle\gamma_1\star\gamma_3,\delta_S,\gamma_2,\gamma_4\rangle_{0,n+3,d}-\langle\gamma_1\star\gamma_2,\delta_S,\gamma_3,\gamma_4\rangle_{0,n+3,d}$$

$$I_1(n)=\sum_{\gamma_2\rightleftarrows\gamma_3}\mathrm{Sign}(\gamma_2,\gamma_3)\sum_{\substack{A\sqcup B=S(n)\\ A,B\neq\emptyset,i=0,d}}\Big(\langle\gamma_1,\gamma_3,\delta_A,\mu\rangle_{0,|A|+3,d-i}\eta^{\mu,\nu}\langle\nu,\delta_B,\gamma_2,\gamma_4\rangle_{0,n+3-|A|,i}\Big)$$

$$I_2(n)=\sum_{\gamma_2\rightleftarrows\gamma_3}\mathrm{Sign}(\gamma_2,\gamma_3)\sum_{\substack{A\sqcup B=S(n)\\ 0<i<d}}\langle\gamma_1,\gamma_3,\delta_A,\mu\rangle_{0,|A|+3,d-i}\eta^{\mu,\nu}\langle\nu,\delta_B,\gamma_2,\gamma_4\rangle_{0,n+3-|A|,i}$$

$$\begin{aligned}I_3(n)=\sum_{\gamma_2\rightleftarrows\gamma_3}\mathrm{Sign}(\gamma_2,\gamma_3)\Big(&\langle\gamma_1,\gamma_3,\mu\rangle_{0,3,d}\eta^{\mu,\nu}\langle\nu,\delta_S,\gamma_2,\gamma_4\rangle_{0,n+3,0}\\ &+\langle\gamma_1,\gamma_3,\delta_S,\mu\rangle_{0,n+3,0}\eta^{\mu,\nu}\cdot\langle\nu,\gamma_2,\gamma_4\rangle_{0,3,d}\Big)\end{aligned}$$

Note that for $d=0$, the WDVV equation is modified to be

$$\begin{aligned}\langle\gamma_1,\gamma_2,\delta_S,\gamma_3\star\gamma_4\rangle_{0,n+3,0}=&\,I_1(n)+\langle\gamma_1,\gamma_3,\delta_S,\gamma_2\star\gamma_4\rangle_{0,n+3,0}\\ &+\langle\gamma_1\star\gamma_3,\delta_S,\gamma_2,\gamma_4\rangle_{0,n+3,0}-\langle\gamma_1\star\gamma_2,\delta_S,\gamma_3,\gamma_4\rangle_{0,n+3,0}\end{aligned} \tag{2.17}$$

Once we have more than two non-primitive insertions, we can choose $\gamma_1, \gamma_2, \gamma_3 \star \gamma_4$ to be these, where $\gamma_4 = \gamma_3^i$ for some $1 \leq i \leq |\gamma_3| - 2$, where $|\gamma_3|$ is the order of γ_3. If there are other nonprimitive insertions with different fixed points, we can choose γ_1, γ_2 to be these insertions. Otherwise, we choose the smallest degree nonprimitive insertion to be γ_3^{i+1}.

1. For $\mathbb{P}^1_{3,3,3}$, each primitive insertion has degree $1/3$, each term in I_0 either vanishes or has an insertion $\mathcal{P}$.

2. For $\mathbb{P}^1_{4,4,2}$, each primitive insertion has degree $1/4$.

3. For $\mathbb{P}^1_{6,3,2}$, each primitive insertion has degree $1/3$, or $1/6$. For the $1/3$ case, it is the same as in $\mathbb{P}^1_{3,3,3}$.

Recursion for genus-0 3-point and 4-point basic correlators

In this subsection, we give an algorithm for the reconstruction of all genus-0 3-point and 4-point basic correlators. For the explicit recursion formulas of various WDVV equations and how the recursion works, see the Appendix. First, we classify all these correlators into six types. Here $\alpha, \beta, \gamma, \xi$ are all primitive elements:

1. $\langle \alpha, \beta^j, \gamma, \xi^{i+1} \rangle_{0,4,d}$, $i, j \geq 1$, supports are not the same point.

2. $\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle_{0,3,d}$.

3. $\langle \gamma, \gamma, \gamma', \gamma' \rangle_{0,4,d}$, $|\gamma|$ is greatest among all primitive elements.

4. $\langle \alpha, \beta^i, \beta^j \rangle_{0,3,d}$, $\alpha \neq \beta$.

5. $\langle \beta, \beta, \beta', \beta' \rangle_{0,4,d}$, $|\beta| = 3$ in case of $\mathbb{P}^1_{6,3,2}$ or $|\beta| = 2$.

6. $\langle \alpha, \alpha^i, \alpha^j \rangle_{0,3,d}$.

Now we start to reconstruct the genus-0 4-point correlators with degree 0,

Lemma II.8. *For Type 1 correlators, WDVV equation implies*

$$\langle \alpha, \beta^j, \gamma, \xi^{i+1} \rangle_{0,4,0} = 0. \tag{2.18}$$

Type 3 correlators $\langle \gamma, \gamma, \gamma', \gamma' \rangle_{0,4,0}$ *can be reconstructed from Chen-Ruan product and* $\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle_{0,3,1}$,

$$\langle \gamma, \gamma, \gamma', \gamma' \rangle_{0,4,0} = -|\gamma|^{-2}. \tag{2.19}$$

Proof. For Type 1 correlator, if there are three primitive insertions, i.e. $j = 1$, then it is either $\langle \Delta_{3,1}, \Delta_{2,1}, \Delta_{2,1}, \Delta_{1,4} \star \Delta_{1,1} \rangle_{0,4,0}^{\mathbb{P}^1_{2,3,6}}$, or $\langle \Delta_{3,1}, \Delta_{3,1}, -, \xi^i \star \xi \rangle_{0,4,0}^{\mathcal{X}}$. Then applying WDVV equation (2.17), they will vanish.

For other cases, i.e. $j \geq 2$, we can assume $j > i$ if $\beta = \xi$. According to dimension axiom, we will always have

$$\deg \alpha \geq \deg \gamma, \alpha \neq \xi. \tag{2.20}$$

We apply WDVV equation (2.17) for $\langle \alpha, \beta^j, \gamma, \xi^i \star \xi \rangle_{0,4,0}$. On the right hand side of the equation, the second term vanishes because $\alpha \neq \xi$. The first term is $\langle \alpha \star \beta^j, \gamma, \xi^i, \xi \rangle_{0,4,0}$, it also vanishes. Or else we must have $\alpha = \beta$ and $(j+1) \deg \alpha < 1$. However, $\deg \alpha \geq \deg \gamma$, which implies $\deg \alpha + \deg(\beta^j) + \deg \gamma \leq 1$. This contradicts with dimension axiom for nonvanishing correlators. The last term will either vanish or equal to $\langle \alpha, \xi^{j+1}, \gamma, \xi^i \rangle_{0,4,0}$, we can continue to apply (2.17) again and again, unless the second insertion is $\mathcal{P}$ or the last insertion is primitive, both correlators are zero.

For Type 3 correlator $\langle \gamma, \gamma, \gamma', \gamma' \rangle_{0,4,0}$, let α, β be the other two primitive insertions and we apply WDVV equation (2.16) to $\langle \gamma, \gamma, \gamma', \gamma' \rangle_{0,4,d}$ for $d = 1$. The equation (2.19) follows from divisor axiom, equation (2.12) and (2.18).

□

Now let us discuss the reconstruction for basic correlators.

Lemma II.9. *All basic correlators are reconstructable for $d \geq 1$.*

Proof. • For Type 1 correlators, go through the proof of Lemma II.8, take any degree $d > 0$, the reconstruction follows.

• For Type 2, if $d > 0$, we consider $\langle \gamma', \gamma', \gamma, \gamma, \beta' \star \beta \rangle_{0,5,d}$, $\beta \neq \gamma$. As $\deg \gamma \leq \deg \beta$, we have $\gamma' \star \gamma' = \gamma' \star \beta' = \gamma' \star \beta = 0$. Lemma II.8 implies $I_3(2)$ also vanish under this choice. The reconstruction follows by

$$\langle \gamma', \gamma', \gamma, \gamma \rangle_{0,4,d} = \frac{|\beta|}{d} \langle \gamma', \gamma', \gamma, \gamma, \beta' \star \beta \rangle_{0,5,d} = \frac{|\beta|}{d} \Big(I_1(2) + I_2(2) \Big). \tag{2.21}$$

• For Type 3 case, for $d > 1$, we consider $\langle \alpha, \beta, \gamma, \gamma \star \gamma' \rangle_{0,4,d}$, where γ has the greatest order among all primitive elements. In this case, $I_0(1)$ and $I_1(1)$ both vanish. Thus Lemma II.8 implies

$$I_3(1) = -\langle \alpha, \beta, \gamma \rangle_{0,3,d} \eta^{\gamma,\gamma'} \langle \gamma', \gamma, \gamma, \gamma' \rangle_{0,4,0} = |\gamma|^{-1} \langle \alpha, \beta, \gamma \rangle_{0,3,d}.$$

Thus we have

$$\langle \alpha, \beta, \gamma \rangle_{0,3,d} = \frac{|\gamma|}{(d-1)} I_2(1). \tag{2.22}$$

• For Type 4 case, we can first reduce to the case of $\langle \alpha, \beta, \beta^i \rangle_{0,3,d}$, $d > 0$. Now choose γ the rest primitive element and apply (2.16) to $\langle \alpha, \beta^i, \beta, \gamma' \star \gamma \rangle_{0,4,d}$. Then $I_0(1), I_1(1)$ and $I_3(1)$ all vanish. Thus the reconstruction follows by

$$\langle \alpha, \beta, \beta^i \rangle_{0,3,d} = \frac{|\gamma|}{d} I_2(1). \tag{2.23}$$

• For Type 5, for $d > 0$, by induction, we already know $\langle \alpha, \gamma, \beta \rangle_{0,3,d+1}$ and Type 1 correlators with degree d are recontructable. Now we apply (2.16) to $\langle \alpha, \gamma, \beta, \beta \star \beta' \rangle_{0,4,d}$. Except for $\langle \alpha, \gamma, \beta \rangle_{0,3,1} \eta^{\beta,\beta'} \langle \beta', \beta, \beta, \beta' \rangle_{0,4,d}$, we already know all the terms in the equality are reconstructable. This gives the recursion for $\langle \beta', \beta, \beta, \beta' \rangle_{0,4,d}$.

- For Type 6, it is the same if we can reconstruct $\langle\alpha,\alpha^i,\alpha^j,\mathcal{P}\rangle_{0,4,d}$. Like we did for the basic correlators, we reduce to the case $\langle\mathcal{P},\alpha,\alpha,\alpha^j,\mathcal{P}\rangle_{0,5,d}$. We can choose β such that $\deg\alpha^i+\deg\beta'\geq 1$. Those terms $I_0(2),I_1(2),I_3(2)$ in formula (2.16) under this choice all vanish ,

$$
\begin{aligned}
\frac{d^2}{|\beta|}\langle\alpha^j,\alpha,\alpha\rangle_{0,3,d} =&\langle\mathcal{P},\alpha^j,\alpha,\alpha,\beta\star\beta'\rangle_{0,5,d}\\
=&2\sum_{i=1}^{d-1}(d-i)\Big(\langle\beta,\alpha,\mu\rangle_{0,3,d-i}\eta^{\mu,\nu}\langle\nu,\alpha,\beta',\alpha^j\rangle_{0,4,i}\\
&-\langle\alpha^j,\alpha,\mu\rangle_{0,3,d-i}\eta^{\mu,\nu}\langle\nu,\alpha,\beta,\beta'\rangle_{0,4,i}\Big).
\end{aligned}
\tag{2.24}
$$

□

Genus-0 resconstruction

Now Theorem II.3 follows from the next lemma.

Lemma II.10. *The WDVV equation and the divisor equation imply that all the genus-0 correlators for $\mathbb{P}^1_{3,3,3},\mathbb{P}^1_{4,4,2},\mathbb{P}^1_{6,3,2}$ are uniquely determined by the pairing, the genus-0 3-point and 4-point correlators.*

Proof. Let us denote by P the maximum complex degree of any primitive class, and by Q the maximum complex degree of any homogeneous non-divisor class. Similarly, as we did for FJRW theory, we can use WDVV, plus the string equation and divisor equation to reconstruct genus-0 primary correlators from the Chen-Ruan product structural constants and basic correlators.

Now let $\langle\gamma_1,\cdots,\gamma_n\rangle^{\mathcal{X}}_{0,n,d}$ be a basic correlator such that the first $n-2$ insertions are primitive. Thus, $\deg\gamma_i\leq P$ for $i\leq n-2$ and $\deg\gamma_{n-1},\deg\gamma_n\leq Q$. By the dimension counting,

$$n-2\leq(n-2)P+2Q.$$

It is easy to obtain the data P, Q for each orbifold:

$$\mathbb{P}^1_{3,3,3} : P = \frac{1}{3}, Q = \frac{2}{3}. \qquad \mathbb{P}^1_{4,4,2} : P = \frac{1}{2}, Q = \frac{3}{4}. \mathbb{P}^1_{6,3,2} : P = \frac{1}{2}, Q = \frac{5}{6}.$$

Thus we have $n = 4$, for $\mathbb{P}^1_{3,3,3}$, $n = 5$ for $\mathbb{P}^1_{4,4,2}$ and $\mathbb{P}^1_{6,3,2}$. We list all the basic genus-0 five-point correlators. Since some of the orbifold points are symmetric, the nonvanishing correlators are same as $\langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{2,1}, \Delta_{1,5}, \Delta_{1,5} \rangle^{\mathbb{P}^1_{6,3,2}}_{0,5,d}$, or $\langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{3,1}, \alpha, \beta \rangle^{\mathcal{X}}_{0,5,d}$, where

$$(\alpha, \beta) = \begin{cases} (\Delta_{1,3}, \Delta_{1,3}), (\Delta_{1,3}, \Delta_{2,3}), & \mathcal{X} = \mathbb{P}^1_{4,4,2}; \\ \\ (\Delta_{1,5}, \Delta_{1,4}), (\Delta_{1,5}, \Delta_{2,2}), & \mathcal{X} = \mathbb{P}^1_{6,3,2}. \end{cases}$$

It follows by applying WDVV (2.16), that all the correlators above can be reconstructed from genus-0 correlators with less than five insertions by choosing some $\gamma_i, i = 1, 2, 3, 4$. For example, for $\langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{3,1}, \Delta_{1,3}, \Delta_{2,3} \rangle^{\mathbb{P}^1_{4,4,2}}_{0,5,d}$, we can choose $\gamma_1 = \Delta_{1,2}, \gamma_2 = \Delta_{1,1}$, $\gamma_3 = \Delta_{2,3}$, $\gamma_4 = \Delta_{3,1}$. □

Fourier series of basic correlators

Let us conclude this subsection with a computational observation. The non-zero, genus-0, 3-point correlators can be expanded as Fourier series. Let us list the first few terms of their Fourier series. For $\mathbb{P}^1_{3,3,3}$, a set of Fourier series are

$$\begin{cases} \langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle_{0,3} = q + q^4 + 2q^7 + 2q^{13} + \cdots \\ \langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{1,1} \rangle_{0,3} = \frac{1}{3} + 2q^3 + 2q^9 + 2q^{12} + \cdots , \end{cases}$$

For $\mathbb{P}^1_{4,4,2}$, a set of Fourier series are

$$\begin{cases} \langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle_{0,3} = q + 2q^5 + q^9 + 2q^{13} + \cdots \\ \langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{1,2} \rangle_{0,3} = \frac{1}{4} + q^4 + q^8 + q^{16} + \cdots \\ \langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{2,2} \rangle_{0,3} = q^2 + 2q^{10} + q^{18} + \cdots , \end{cases}$$

For $\mathbb{P}^1_{6,3,2}$, a set of Fourier series are

$$\begin{cases} \langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{1,4} \rangle_{0,3} = \dfrac{1}{6} + q^6 + q^{18} + q^{24} + \cdots \\ \langle \Delta_{1,1}, \Delta_{1,2}, \Delta_{1,3} \rangle_{0,3} = \dfrac{1}{6} + q^{12} + \cdots \\ \langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle_{0,3} = q + 2q^7 + 2q^{13} + 2q^{19} + \cdots \\ \langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{2,2} \rangle_{0,3} = q^2 + q^8 + 2q^{14} + \cdots \\ \langle \Delta_{1,1}, \Delta_{1,2}, \Delta_{3,1} \rangle_{0,3} = q^3 + q^9 + 2q^{21} + \cdots \\ \langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{1,3} \rangle_{0,3} = q^4 + q^{16} + 2q^{28} + \cdots \end{cases}$$

After the discussion of modularity properties in Chapterr VIII, we can easily see in each case, the listed Fourier series forms a basis of the vector space of modular forms of weight 1, with the modular group $\Gamma(3), \Gamma(4)$ and $\Gamma(6)$ in respective cases.

2.5 Convergence

Let $\mathscr{S}$ be the set of generators of $H^*_{\mathrm{CR}}(\mathcal{X})$ introduced in (2.11). We define

$$I^{GW}_{g,n,d} := \max_{\alpha_i \in \mathscr{S}} \left| \langle \alpha_1, \cdots, \alpha_n \rangle_{g,n,d} \right|. \tag{2.25}$$

Here is the main estimation in this section. The proof will be given in Chapter IX.

Theorem II.11. *Let us assume $\alpha_i \in \mathscr{S}$, and $\alpha_i \neq \mathcal{P}$ for $l_i = 0$. Let us denote $\chi := 2g-2+n$, $L = \sum_i l_i$. Then for $\chi \geq 0$, we have:*

$$\left| \langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n) \rangle_{g,n,d} \right| \leq \begin{cases} d^{\chi-2} C(\chi)^{\chi+(g+L+1)d-2}, & if\, d > 0. \\ C(\chi)^{\chi-1}, & if\, d = 0. \end{cases} \tag{2.26}$$

Here $C(\chi)$ is a sufficient large constant which depends increasingly only on χ.

Now we prove the convergence of the Gromov-Witten part in Theorem I.4.

Proof. For $\mathbf{t} = s\mathcal{P}$, recall the Divisor equation for the ancestor correlators (2.8), we have the estimation:

$$
\begin{aligned}
&\Big|\langle\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n)\rangle\rangle_{g,n}(s\mathcal{P})\Big| \\
&= \Big| \sum_{d\geq 0}\sum_{k\geq 0} \frac{1}{k!}\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n), s\mathcal{P}, \cdots, s\mathcal{P}\rangle_{g,n+k,d}\Big| \\
&\leq \Big|\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n)\rangle_{g,n,0}\Big| + \Big| \sum_{d\geq 1}\sum_{k\geq 0} \frac{1}{k!}\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n)\rangle_{g,n,d} s^k d^k\Big| \\
&\leq \sum_{d\geq 0} \Big|e^s\Big|^d d^{\chi-2} C(\chi)^{\chi+(g+L+1)d-2}.
\end{aligned}
$$

This is convergent for $\Big|e^s\, C(\chi)^{g+L+1}\Big| \leq 1/2$.

Now we consider $\mathbf{t} = s\mathcal{P} + \sum_{i\geq 0} t_i\phi_i$, where ϕ_i ranges over the homogeneous basis other than $\mathcal{P}$. The dimension formula (2.7) implies the function $\langle\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n)\rangle\rangle_{g,n}(\mathbf{t})$ is a polynomial of t_i with coefficients are ancestor functions valued at $s\mathcal{P}$. As the number of terms of the monomials in this polynomial depends only on the genus g and number of marked points n. It follows that for g, n fixed, $\langle\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n)\rangle\rangle_{g,n}(\mathbf{t})$ is convergent near $\mathrm{Re}(s) \ll 0, t_i = 0$ for $i \geq 0$. □

CHAPTER III

Fan-Jarvis-Ruan-Witten theory

3.1 Introduction

For any non-degenerate, quasi-homogeneous polynomial W with N variables, Fan, Jarvis and Ruan, following a suggestion of Witten, introduced a family of moduli spaces and constructed a virtual fundamental cycle. The latter gives rise to a cohomological field theory, which is now called the FJRW theory. Let us briefly review the FJRW theory only for the group G_W. We refer to [FJR2] for general cases and more details.

Recall the *group of diagonal symmetries* G_W of the polynomial W is

$$G_W := \left\{(\lambda_1, \ldots, \lambda_N) \in (\mathbb{C}^*)^N \middle| W(\lambda_1 X_1, \ldots, \lambda_N X_N) = W(X_1, \ldots, X_N)\right\}.$$

The *FJRW state space* $\mathscr{H}_{W,G_W}$ (or $\mathscr{H}_W$ for short) is the direct sum of all G_W-invariant relative cohomology:

$$\mathscr{H}_W := \bigoplus_{h \in G_W} H_h, \quad H_h := H^*(\mathbb{C}_h; W_h^\infty; \mathbb{C})^{G_W}. \tag{3.1}$$

Here $\mathbb{C}_h (h \in G_W)$ is the h-invariant subspace of $\mathbb{C}^N$, W_h is the restriction of W to $\mathbb{C}_h$, $\mathrm{Re} W_h$ is the real part of W_h, and $W_h^\infty = (\mathrm{Re} W_h)^{-1}(M, \infty)$, for some $M \gg 0$.

The vector space $H_h (h \in G_W)$ has a natural grading given by the degree of the relative cohomology classes. However, for the purposes of the FJRW theory we need a modification of the standard grading. Namely, for a given homogeneous element $\alpha \in H_h$ we define

$$\deg_W \alpha := \deg \alpha + \sum_{i=1}^{N} (\Theta_i^h - q_i),$$

where $\deg \alpha$ is the cohomology degree of α and the numbers $\Theta_i^h \in [0,1) \cap \mathbb{Q}$ are such that

$$h = \left(\mathbf{e}[\Theta_1^h], \ldots, \mathbf{e}[\Theta_N^h]\right) \in (\mathbb{C}^*)^N,$$

where for $y \in \mathbb{R}$, we put $\mathbf{e}[y] := \exp(2\pi\sqrt{-1}y)$. Clearly the numbers Θ_i^h are uniquely determined from h. For any $\alpha \in H_h$, we define

$$\Theta(\alpha) := \left(\mathbf{e}[\Theta_1^h], \ldots, \mathbf{e}[\Theta_N^h]\right). \tag{3.2}$$

The elements in H_h are called *narrow* (resp. *broad*) and H_h is called a *narrow sector* (resp. *broad sector*) if $\mathbb{C}_h = \{0\}$ (resp. $\mathbb{C}_h \neq \{0\}$). For invertible simple elliptic singularities, the space $H^*(\mathbb{C}_h; W_h^\infty; \mathbb{Q})$ is one-dimensional for all narrow sectors H_h. We always choose a generator $\alpha \in H_h$ such that

$$\alpha := 1 \in H^*(\mathbb{C}_h; W_h^\infty; \mathbb{Q}). \tag{3.3}$$

In general, in order to describe the broad sectors, we have to represent the relative cohomology classes by differential forms; then there is an identification

$$(\mathscr{H}_{W,G}, \langle\ ,\ \rangle) \equiv \left(\bigoplus_{h \in G} (\mathscr{Q}_{W_h} \omega_h)^G, \mathrm{Res} \right), \tag{3.4}$$

where ω_h is the restriction of the standard volume form to the fixed locus $\mathbb{C}_h$, Res is the residue pairing, and $\langle\ ,\ \rangle$ is a non-degenerate pairing induced from the intersection of relative homology cycles. There exists a basis of the narrow sectors such that the pairing $\langle v_1, v_2 \rangle$, $v_i \in H_{h_i}$, is 1 if $h_1 h_2 = 1$ and 0 otherwise. The vectors in the broad sectors are orthogonal to the vectors in the narrow sectors. In order to compute the pairing on the broad sectors one needs to use the identification (3.4) and compute an appropriate residue pairing. In our case however, we can express all invariants using narrow sectors only. So a more detailed description of the broad sectors is not needed. We refer to [FJR2] for more details.

W-spin structure

Let (W, G) be an admissible pair. A *W-spin structure* on a genus-g stable curve C with n marked orbifold points $(z_1, \ldots, z_n)$ is a collection of N (N is the number of variables in W) orbifold line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_N$ on C and isomorphisms

$$\psi_a : M_a(\mathcal{L}_1, \ldots, \mathcal{L}_N) \to \omega_C(-z_1 - \cdots - z_n),$$

where ω_C is the dualizing sheaf on C and M_a are the homogeneous monomials whose sum is W. The orbifold line bundles have a monodromy near each marked point z_i which determines an element $h_i \in G$. In particular, if H_{h_i} is a narrow (resp. broad) sector we say that the marked point is narrow (resp. broad). For fixed g, n, and $h_1, \ldots, h_n \in G$, Fan-Jarvis-Ruan (see [FJR2]) constructed the compact moduli space $\mathscr{W}_{g,n}(h_1, \cdots, h_n)$ of nodal stable curves equipped with a W-spin structure. In this compactification the line bundles $(\mathcal{L}_1, \ldots, \mathcal{L}_N)$ are allowed to be orbifold at the nodes in such a way that the monodromy around each node is an element of G as well. The moduli space has a decomposition into a disjoint union of moduli subspaces $\mathscr{W}_{g,n}(\Gamma_{h_1,\ldots,h_n})$ consisting of W-spin structures on curves C whose dual graph is $\Gamma_{h_1,\ldots,h_n}$. Recall that the dual graph of a nodal curve C is a graph whose vertices are the irreducible components of C, edges are the nodes, and tails are the marked points. The latter are decorated by elements $h_i \in G$, so the tails of our graphs are also colored respectively. We omit the subscript $(h_1, \ldots, h_n)$ whenever the decoration is understood from the context. The connected component $\mathscr{W}_{g,n}(\Gamma_{h_1,\ldots,h_n})$ is naturally stratified by fixing the monodromy transformations around the nodes, i.e., the strata are in one-to-one correspondence with the colorings of the edges of the dual graph $\Gamma_{h_1,\cdots,h_n}$.

Fan–Jarvis–Ruan constructed a virtual fundamental cycle $\left[\mathscr{W}_{g,n}(\Gamma)\right]^{\mathrm{vir}}$ of $\mathscr{W}_{g,n}(\Gamma)$ (see

[FJR2]), which gives rise to a CohFT

$$\Lambda^{W,G}_{g,n} : (\mathscr{H}_{W,G})^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}).$$

For brevity, we put $\Lambda^{W}_{g,n}$ for $\Lambda^{W,G_W}_{g,n}$.

Axioms for simple elliptic singularities

Let us list some general properties of the FJRW correlators of a simple elliptic singularity W. Here $N = 3$. See [FJR2, FJR3] for the proofs.

- (Selection rule) If the correlator $\langle \tau_{k_1}(\alpha_1), \ldots, \tau_{k_n}(\alpha_n)\rangle^W_{g,n}$ is non-zero; then

$$\sum_{i=1}^{n} \deg_W(\alpha_i) + \sum_{i=1}^{n} k_i = 2g - 2 + n. \tag{3.5}$$

- (Line bundle criterion). If the moduli space $\mathscr{W}_{g,n}(h_1, \ldots, h_n)$ is non-empty, then the degree of the desingularized line bundle $|\mathcal{L}_j|$ is an integer, i.e.

$$\deg(|\mathcal{L}_j|) = q_j(2g - 2 + n) - \sum_{k=1}^{n} \Theta_j^{h_k} \in \mathbb{Z}. \tag{3.6}$$

- (Index zero). If $\dim \mathscr{W}_{g,n}(\Gamma_{h_1,\ldots,h_n}) = 0$ and all the decorations on marked points are narrow. If $\pi_* \left(\bigoplus_{i=1}^3 \mathcal{L}_i\right)$ and $R^1\pi_* \left(\bigoplus_{i=1}^3 \mathcal{L}_i\right)$ are both vector bundles of the same rank. We denote the Witten map $\mathscr{D}_{\rm wit}\colon (X_1, \cdots, X_N) \mapsto (\frac{\overline{\partial W}}{\partial X_1}, \cdots, \frac{\overline{\partial W}}{\partial X_N})$, then

$$\left[\mathscr{W}(\Gamma_{h_1,\ldots,h_n})\right]^{\rm vir} = \deg(\mathscr{D}_{\rm wit}) \left[\mathscr{W}(\Gamma_{h_1,\ldots,h_n})\right]. \tag{3.7}$$

- (Concavity) Suppose that all the decorations on marked points are narrow, π is the morphism from the universal curve to $\mathscr{W}_{g,n}(h_1, \ldots, h_n)$ and $\pi_* \left(\bigoplus_{i=1}^3 \mathcal{L}_i\right) = 0$ holds; then

$$[\mathscr{W}_{g,n}(h_1, \ldots, h_n)]^{\rm vir} = c_{\rm top}\left(-R^1\pi_* \bigoplus_{i=1}^{3} \mathcal{L}_i\right) \cap [\mathscr{W}_{g,n}(h_1, \ldots, h_n)]. \tag{3.8}$$

Orbifold Grothendieck-Riemann-Roch

Let $\alpha_i = 1 \in H_{h_i}$, $1 \leq i \leq 4$ be the generators (cf. (3.3)). The concavity formula (3.8) implies that $\Lambda^W_{0,4}(\alpha_1, \ldots, \alpha_4) \in H^*(\overline{\mathcal{M}}_{0,4}, \mathbb{C})$. According to the orbifold Grothendieck-Riemann-Roch formula (see [Chi], Theorem 1.1.1) $\Lambda^W_{0,4}(\alpha_1, \ldots, \alpha_4)$ is

$$\sum_{i=1}^{3} \left(\frac{B_2(q_i)}{2} \kappa_1 - \sum_{j=1}^{4} \frac{B_2(\Theta_i^{h_j})}{2} \psi_j + \sum_{\Gamma \in \mathbf{\Gamma}_{0,4,W}(h_1,\ldots,h_4)} \frac{B_2(\Theta_i^{h_\Gamma})}{2} [\Gamma] \right), \tag{3.9}$$

where B_2 is the second Bernoulli polynomial

$$B_2(y) = y^2 - y + \frac{1}{6},$$

$[\Gamma]$ is the boundary class on $\overline{\mathcal{M}}_{g,n}$ corresponding to the graph Γ, and $\mathbf{\Gamma}_{0,4,W}(h_1, \ldots, h_4)$ is the set of graphs with one edge decorated by G_{W^T}. The graph Γ has 4 tails decorated by h_1, h_2, h_3, h_4 and its edge is decorated by h_Γ. Due to (3.6), the moduli space $\mathscr{W}_{0,4}(h_1, \ldots, h_4)$ is non-empty only if h_Γ satisfies an appropriate constraint involving $h_1, \ldots, h_4$. It is easy to see that the formula does not depend on the choice of h_Γ.

3.2 Simple elliptic singularities and their FJRW rings

Let W^T be the Berglund-Hübsch-Krawitz mirror of W, see Chapter V. From now on, we will consider W as in Table (1.1). Then up to symmetry, we can still consider W^T as an element in Table (1.1). However, W and W^T may belong to different column. We will consider the FJRW theory for (W^T, G_{W^T}) and the Saito-Givental theory for W.

Since $\mathscr{H}_{W^T} := \mathscr{H}_{W^T, G_{W^T}}$ is the state space of a CohFT, it has a Frobenius algebra structure, where the multiplication $\bullet$ is defined as follows:

$$\langle \alpha_1 \bullet \alpha_2, \alpha_3 \rangle^{W^T} = \langle \alpha_1, \alpha_2, \alpha_3 \rangle^{W^T}_{0,3}.$$

For ISES the product $\bullet$ (when $G = G_{W^T}$ is maximal) was computed by M. Krawitz (see [Kr]). More precisely, he constructed a basis of $\mathscr{H}_{W^T}$, which gives rise to an isomorphism between the Frobenius algebra $\mathscr{H}_{W^T}$ and the Jacobi algebra $\mathscr{Q}_W$.

Now we give an explicit description of the generators. For a general description of the ring morphism $\mathscr{H}_{W^T} \to \mathscr{Q}_W$, we refer the interested reader to [Kr]. Since some of our ISESs are equivalent to 2-variable singularities of chain type or of loop type, we also refer to [FS] and [Ac] for those particular examples.

For every ISES W^T, there exists a vector $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$ such that

$$G_{W^T} \cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \mathbb{Z}/n_3\mathbb{Z}.$$

In particular, we can identify the vector $\mathbf{n}$ with the group G_{W^T}. We assume $n_1 \geq n_2 \geq n_3$ and omit those $n_i = 1$ in $\mathbf{n}$. For example, if $W^T = X_1^3 + X_1X_2^4 + X_3^2$, then

$$G_{W^T} = \left\{(\lambda_1, \lambda_2, \lambda_3)\middle|\lambda_1^3 = \lambda_1\lambda_2^4 = \lambda_2^2 = 1\right\} \cong (12, 2).$$

Let $h = (i, j, k) \in G_{W^T}$. If

$$1 \leq i < n_1, \quad 1 \leq j < n_2, \quad 1 \leq k < n_3,$$

then H_h is a one-dimensional space of narrow sectors. Let

$$\mathbf{e}_{i,j,k} := 1 \in H^0(\mathbb{C}_h; W_h^\infty; \mathbb{Q}).$$

Example III.1. We compute the FJRW ring for loop singularity W^T, with $W \in E_6^{(1,1)}$.

$$W^T = X_1^2X_3 + X_1X_2^2 + X_2X_3^2, \quad G_{W^T} = \left\{\mathbf{e}_i = \left(\mathbf{e}[\frac{i}{9}], \mathbf{e}[\frac{4i}{9}], \mathbf{e}[-\frac{2i}{9}]\right), i = 1, \dots, 8\right\} \cong \mathbb{Z}/8\mathbb{Z}.$$

All nonzero 3-point genus-0 correlators are

$$\begin{cases} \langle \mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_1 \rangle_{0,3} = \langle \mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_4 \rangle_{0,3} = \langle \mathbf{e}_7, \mathbf{e}_7, \mathbf{e}_7 \rangle_{0,3} = -2; \\ \\ \langle \mathbf{e}_3, \mathbf{e}_i, \mathbf{e}_{9-i} \rangle_{0,3} = \langle \mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_7 \rangle_{0,3} = 1. \end{cases}$$

The first row uses Index Zero Axiom (3.7) and the second row uses Concavity Axiom (3.8). It is easy to see $\mathbf{e}_3$ is the identity element and the ring relations are

$$2\,\mathbf{e}_1 \bullet \mathbf{e}_4 + \mathbf{e}_7^2 = 2\,\mathbf{e}_4 \bullet \mathbf{e}_7 + \mathbf{e}_1^2 = 2\,\mathbf{e}_7 \bullet \mathbf{e}_1 + \mathbf{e}_4^2 = 0.$$

Thus we obtain a ring isomorphism between $\mathscr{H}_{W^T}$ and $\mathscr{Q}_W$:

$$\rho_1 = \mathbf{e}_4 \mapsto X_1, \quad \rho_2 = \mathbf{e}_1 \mapsto X_2, \quad \rho_3 = \mathbf{e}_7 \mapsto X_3.$$

For all 13 types of ISESs with a maximal admissible group, there is a unique narrow sector ρ_{-1}, with $\deg_{W^T}(\rho_{-1}) = 1$ and

$$\Theta(\rho_{-1}) := \left(1 - q_1^T, 1 - q_2^T, 1 - q_3^T\right).$$

There are 13 types of ISESs, but only for 9 of them do not have broad generators. The narrow sectors have the advantage that we can use the powerful concavity axiom (3.8). Combined with the remaining properties of the correlators and the WDVV equations this allows us to reconstruct all genus-0 FJRW invariants. According to the reconstruction theorem in [KS], we can also reconstruct the higher genus FJRW invariants, i.e., the total ancestor potential function $\mathscr{A}_{FJRW}^{W^T}$.

In the remaining 4 cases, we can offset the complication of having broad generators only for $W^T = X_1^2 + X_1X_2^2 + X_2X_3^3$. The maximal abelian group is of order 12. Its FJRW vector space has eight generators:

$$\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_9, \mathbf{e}_{11}, R_4, R_8.$$

Here R_4 and R_8 are the cohomology classes represented by the following forms:

$$R_h = dX_1 \wedge dX_2 \in H^2(\mathbb{C}_h; W_h^\infty; \mathbb{Q}), \quad h = 4, 8 \in G_{W^T}.$$

Note that R_4 and R_8 are G_{W^T}-invariant elements in $\mathscr{Q}_{W_h}\omega_h$ where $h \in G_{W^T}$ acts on each factor X_i, such that dX_i is a divisor of ω_h, as multiplication by $\mathbf{e}[q_i^T]$. Although one of the generators (R_4) is broad, we have enough WDVV equations to reconstruct the correlators containing broad sectors from correlators with only narrow sectors and apply the concavity axioms.

For the other three types of ISESs, we can still compute some genus-0 4-point correlators with broad sectors, but we could not reconstruct the complete theory only from correlators with narrow insertions. In other words, for 10 out of the 13 ISESs we can compute all FJRW invariants. We call them *good* cases. These cases and the corresponding generators ρ_i of the FJRW ring $\mathscr{H}_{W^T,G_{W^T}}$ are listed in Table 3.1, 3.2, 3.3 below. Here if $W \in E_\mu^{(1,1)}$, then we say $W^T \in \left(E_\mu^{(1,1)}\right)^T$,

Table 3.1: Generators of the FJRW ring in the case $\left(E_6^{(1,1)}\right)^T$

W^T	G_{W^T}	$\Theta(\mathbf{e}_{i,j,k})$	ρ_1	ρ_2	ρ_3	ρ_{-1}
$X_1^3 + X_2^3 + X_3^3$	$(3,3,3)$	$\mathbf{e}[\frac{i}{3}], \mathbf{e}[\frac{j}{3}], \mathbf{e}[\frac{k}{3}]$	$\mathbf{e}_{2,1,1}$	$\mathbf{e}_{1,2,1}$	$\mathbf{e}_{1,1,2}$	$\mathbf{e}_{2,2,2} = \rho_1\rho_2\rho_3$
$X_1^2X_3 + X_1X_2^2 + X_2X_3^2$	8	$\mathbf{e}[\frac{i}{9}], \mathbf{e}[\frac{4i}{9}], \mathbf{e}[-\frac{2i}{9}]$	$\mathbf{e}_4$	$\mathbf{e}_1$	$\mathbf{e}_7$	$\mathbf{e}_6 = \rho_1\rho_2\rho_3$
$X_1^2 + X_1X_2^2 + X_2X_3^3$	12	$\mathbf{e}[\frac{i}{2}], \mathbf{e}[-\frac{i}{4}], \mathbf{e}[\frac{i}{12}]$	R_4	$\mathbf{e}_1$	$\mathbf{e}_7$	$\mathbf{e}_9 = \rho_2\rho_3^2$

Table 3.2: Generators of the FJRW ring in the case $\left(E_7^{(1,1)}\right)^T$

W^T	G_{W^T}	$\Theta(\mathbf{e}_{i,j,k})$	ρ_1	ρ_2	ρ_{-1}
$X_1^4 + X_2^4 + X_3^2$	$(4,4,2)$	$\mathbf{e}[\frac{i}{4}], \mathbf{e}[\frac{j}{4}], \mathbf{e}[\frac{1}{2}]$	$\mathbf{e}_{2,1}$	$\mathbf{e}_{1,2}$	$\mathbf{e}_{2,2} = \rho_1^2\rho_2^2$
$X_1^3X_2 + X_1X_2^3 + X_3^2$	$(8,2)$	$\mathbf{e}[-\frac{3i}{8}], \mathbf{e}[\frac{i}{8}], \mathbf{e}[\frac{1}{2}]$	$\mathbf{e}_1$	$\mathbf{e}_5$	$\mathbf{e}_6 = \rho_1^2\rho_2^2$
$X_1^3 + X_1X_2^4 + X_3^2$	$(12,2)$	$\mathbf{e}[\frac{-i}{3}], \mathbf{e}[\frac{i}{12}], \mathbf{e}[\frac{1}{2}]$	$\mathbf{e}_1$	$\mathbf{e}_5$	$\mathbf{e}_{10} = \rho_1\rho_2^3$
$X_1^3 + X_1X_2^2 + X_2X_3^2$	12	$\mathbf{e}[\frac{i}{3}], \mathbf{e}[-\frac{i}{6}], \mathbf{e}[\frac{i}{12}]$	$\mathbf{e}_5$	$\mathbf{e}_1$	$\mathbf{e}_8 = -\frac{1}{2}\rho_1^3\rho_2$

Table 3.3: Generators of the FJRW ring in the case $\left(E_8^{1,1}\right)^T$

W^T	G_{W^T}	$\Theta(\mathbf{e}_{i,j,k})$	ρ_1	ρ_2	ρ_{-1}
$X_1^6 + X_2^3 + X_3^2$	$(6,3,2)$	$\mathbf{e}[\frac{i}{6}], \mathbf{e}[\frac{j}{3}], \mathbf{e}[\frac{1}{2}]$	$\mathbf{e}_{2,1}$	$\mathbf{e}_{1,2}$	$\mathbf{e}_{5,2} = \rho_1^4\rho_2$
$X_1^3 + X_1X_2^2 + X_3^3$	$(6,3)$	$\mathbf{e}[\frac{-i}{3}], \mathbf{e}[\frac{i}{6}], \mathbf{e}[\frac{j}{3}]$	$\mathbf{e}_{1,1}$	$\mathbf{e}_{2,2}$	$\mathbf{e}_{4,2} = \rho_1\rho_2\rho_3$
$X_1^4 + X_1X_2^3 + X_3^2$	$(12,2)$	$\mathbf{e}[\frac{-i}{4}], \mathbf{e}[\frac{i}{12}], \mathbf{e}[\frac{1}{2}]$	$\mathbf{e}_2$	$\mathbf{e}_7$	$\mathbf{e}_9 = \rho_1^2\rho_2^2$

3.3 Reconstruction

For an ISES W^T, its total ancestor potential $\mathscr{A}^{W^T}$ can be reconstructed from genus-0 primary correlators. The idea is same as what we did for Gromov-Witten theory in the previous chapter. We replace the dimension argument in Gromov-Witten theory by

Selection rule. Here we do not have divisor axiom. However, it is not necessary since we only need the divisor equation to reduce the degree of Gromov-Witten correlators and there is no degree variable here in FJRW theory.

We say that a homogeneous element $\alpha \in \mathscr{H}_{W^T}$ is *primitive* if it cannot be decomposed as a product $a' \bullet a''$ of two elements a' and a'' of non-zero degrees. We also say that a genus-0 correlator is a *basic correlator* if there are at most two non-primitive insertions, neither of which is the identity.

Over all, we have the following statement.

Lemma III.2. *For an invertible simple elliptic singularity W^T the total ancestor FJRW potential of (W^T, G_{W^T}) is reconstructed from the pairing, the FJRW ring structure constants and the 4-point basic correlators with one of the insertions being a top degree element.*

Now let us prove the reconstruction theorem of all genera for the three types of singularities paired with G_W. We again use g-reduction Lemma II.4 for higher genus reconstruction. We use the following notation as in [KS] and consider their transpose singularity for FJRW theory.

$$\begin{cases} P_8 = X_1^3 + X_2^3 + X_3^3 \in E_6^{(1,1)}, \\ \\ X_9 = X_1^3 X_2 + X_2^2 X_3 + X_3^2 \in E_7^{(1,1)}, \\ \\ J_{10} = X_1^3 X_2 + X_2^2 + X_3^3 \in E_8^{(1,1)}. \end{cases}$$

For simplicity, we first state the result for $W = P_8, X_9^T, J_{10}^T$. It can be easily generalized to all other cases. We make the following changes on the notations. For P_8, we let $\mathbf{e}_{x^{i-1}y^{j-1}z^{k-1}} = \mathbf{e}_{i,j,k}$. For J_{10}^T, we let $\mathbf{e}_{6j-6+i} = \mathbf{e}_{i,j}$.

Lemma III.3. *For the three types of elliptic singularities $W = P_8, X_9^T, J_{10}^T$, the correlator $\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n) \rangle_{g,n}^{W,G}$ in FJRW theory is uniquely reconstructed by tautological relations*

(WDVV-equation, Getzler's relation and g-reduction) from genus-0 primary correlators.

Proof. As for these three elliptic singularities, the central charge $\hat{c}_W = 1$, thus the result easily follows from Theorem III.4 here and Theorem 6.2.1 in [FJR2]. $\square$

$g = 1$

In this subsection, we show that the genus-1 primary correlators can be reconstructed from genus-0 primary correlators. By the selection rule (3.5), the nonvanishing genus-1 primary correlators must be of the form $\langle \rho_{-1}, \cdots, \rho_{-1} \rangle_{1,n}$, where $\rho_{-1} = \mathbf{e}_{xyz}, \mathbf{e}_8$ and $\mathbf{e}_{10}$, respectively in the P_8, X_9^T and J_{10}^T cases.

Theorem III.4. *For all simple elliptic singularites with maximal admissible group, the genus-1 FJRW correlators can be reconstructed from genus-0 FJRW correlators by the Getzler relation.*

Proof. P_8**-case:**

In this case, we need to reconstruct $\langle \mathbf{e}_{xyz}, \cdots, \mathbf{e}_{xyz} \rangle_{1,n}^{P_8}, n \geq 2$. We have the forgetful map $\pi_{4,n-2} : \overline{\mathcal{M}}_{1,n+2} \to \overline{\mathcal{M}}_{1,4}$. Let $S = \{1, \cdots, n-2\}$. Thus

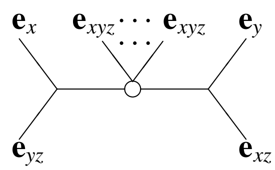
$$\pi_{4,n-2}^{-1}(\Delta_{12,34}) = \sum_{A \cup B \cup C = S} \Delta_{12A,B,C34}.$$

Now we choose $n+2$ insertions: the first four are $\mathbf{e}_x, \mathbf{e}_{yz}, \mathbf{e}_y, \mathbf{e}_{xz}$, the others are $\mathbf{e}_{xyz}$. Integrating the class $\Lambda_{1,n+2}^{P_8}(\mathbf{e}_x, \mathbf{e}_{yz}, \mathbf{e}_y, \mathbf{e}_{xz}, \mathbf{e}_{xyz}, \cdots, \mathbf{e}_{xyz})$ on $\pi_{4,n-2}^{-1}([\Delta_{12,34}])$, we have:

$$\begin{aligned}
&\int_{\pi_{4,n-2}^{-1}([\Delta_{12,34}])} \Lambda_{1,n+2}^{P_8}(\mathbf{e}_x, \mathbf{e}_{yz}, \mathbf{e}_y, \mathbf{e}_{xz}, \mathbf{e}_{xyz}, \cdots, \mathbf{e}_{xyz}) \\
&= \int_{[\Delta_{12,S,34}]} \Lambda_{1,n+2}^{P_8}(\mathbf{e}_x, \mathbf{e}_{yz}, \mathbf{e}_y, \mathbf{e}_{xz}, \mathbf{e}_{xyz}, \cdots, \mathbf{e}_{xyz}) \\
&= \langle \mathbf{e}_x, \mathbf{e}_{yz}, 1 \rangle_{0,3}^{P_8} \eta^{1,\mathbf{e}_{xyz}} \langle \mathbf{e}_{xyz}, \cdots, \mathbf{e}_{xyz} \rangle_{1,n}^{P_8} \eta^{\mathbf{e}_{xyz},1} \langle 1, \mathbf{e}_y, \mathbf{e}_{xz} \rangle_{0,3}^{P_8} \\
&= \langle \mathbf{e}_{xyz}, \cdots, \mathbf{e}_{xyz} \rangle_{1,n}^{P_8}.
\end{aligned}$$

The second equality uses the Splitting Axiom. The first equality is a consequence of the Selection Rule (3.5) and the String equation. The Selection rule requires that each insertion for a non-zero genus-1 primary correlator is $\mathbf{e}_{xyz}$ and the String Equation implies that the genus-0 primary correlator with more than four insertions will vanish if there is one insertion the identity element. Thus the non-zero contribution partition should be $B = S$, and $A = C = \emptyset$. The corresponding decorated dual graph for $\pi^{-1}_{4,n-2}(\Delta_{12,34})$ is

$$\Delta_{12,S,34}(\mathbf{e}_x, \mathbf{e}_{yz}, \mathbf{e}_y, \mathbf{e}_{xz}, \mathbf{e}_{xyz}, \cdots, \mathbf{e}_{xyz}) =$$



Other decorated dual graphs are obtained similarly. As $\delta_{2,2} = [\Delta_{2,2}]$ is the S_4-invariant, we integrate over each stratum and finally get

$$\int_{\pi^{-1}_{4,n-2}(\delta_{2,2})} \Lambda^{P_8}_{1,n+2}(\mathbf{e}_x, \mathbf{e}_{yz}, \mathbf{e}_y, \mathbf{e}_{xz}, \mathbf{e}_{xyz}, \cdots, \mathbf{e}_{xyz}) = 3\langle \mathbf{e}_{xyz}, \cdots, \mathbf{e}_{xyz}\rangle^{P_8}_{1,n}. \tag{3.10}$$

We observe that only $\delta_{2,3}$ can contain at most n insertions for the genus-1 component. However, one of the insertions is decorated with an element from the first four insertions. Thus the integration vanishes according to the selection rule. On the other hand, when we integrate the same class on other dimension two strata in Getzler's relation (2.14), all the genus-1 correlators will have at most $n-1$ insertions. Thus Getzler's relation implies $\langle \mathbf{e}_{xyz}, \cdots, \mathbf{e}_{xyz}\rangle^{P_8}_{1,n}$ $(n \geq 2)$ can be reconstructed from genus-1 correlators with fewer insertions and other genus-0 correlators.

Now we consider the integration of the class $\Lambda^{P_8}_{1,4}(\mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_{xyz})$ on those codimension two strata of $\overline{\mathcal{M}}_{1,4}$. We can discuss similarly as above. The integration on $\delta_{2,2}, \delta_{2,3}, \delta_{2,4}$ will all vanish. However,

$$\begin{aligned}\int_{[\Delta_{1,234}]} \Lambda^{P_8}_{1,4}(\mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_{xyz}) &= \langle \mathbf{e}_{xyz}\rangle^{P_8}_{1,1} \eta^{\mathbf{e}_{xyz},1} \langle \mathbf{e}_1, \mathbf{e}_x, \mathbf{e}_{yz}\rangle^{P_8}_{0,3} \eta^{\mathbf{e}_{yz},\mathbf{e}_x} \langle \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_{xyz}\rangle^{P_8}_{0,4} \\ &= \frac{1}{3}\langle \mathbf{e}_{xyz}\rangle^{P_8}_{1,1}.\end{aligned}$$

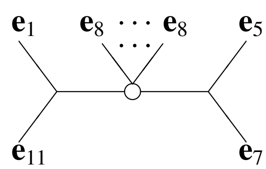
Here we use the fact $\langle \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_{xyz}\rangle^{P_8}_{0,4} = \frac{1}{3}$, will be computed in Lemma III.10. Overall,

$$\int_{\delta_{3,4}} \Lambda^{P_8}_{1,4}(\mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_{xyz}) = \frac{4}{3}\langle \mathbf{e}_{xyz}\rangle^{P_8}_{1,1}.$$

Now applying the Getzler's relation again, other contributions are of genus-0, and $\langle \mathbf{e}_{xyz} \rangle_{1,1}^{P_8}$ can be reconstructed from genus-0 primary correlators.

X_9^T**-case:**

For $n \geq 2$, we choose $n+2$ insertions: the first four are $\mathbf{e}_1, \mathbf{e}_{11}, \mathbf{e}_5, \mathbf{e}_7$, the others are $\mathbf{e}_8$. The nonzero contribution of $\Delta_{12,34}$ comes from the following decorated dual graph:



and

$$\int_{\pi_{4,n-2}^{-1}(\delta_{2,2})} \Lambda_{1,n+2}^{X_9^T}(\mathbf{e}_1, \mathbf{e}_{11}, \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_8, \cdots, \mathbf{e}_8) = 3\langle \mathbf{e}_8, \cdots, \mathbf{e}_8 \rangle_{1,n}^{X_9^T}. \tag{3.11}$$

The integrations of $\Lambda_{1,n+2}^{X_9^T}(\mathbf{e}_1, \mathbf{e}_{11}, \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_8, \cdots, \mathbf{e}_8)$ on other strata in Getzler's relation only produce genus-1 correlators with lower insertions and genus-0 correlators.

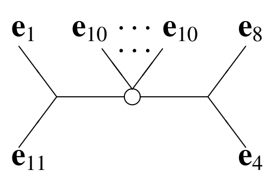
For $n = 1$, we integrate $\Lambda_{1,4}^{X_9^T}(\mathbf{e}_1, \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_7)$ on the Getzler's relation. It vanishes on strata with genus-1 component except for $\delta_{3,4}$. We have

$$\int_{\delta_{3,4}} \Lambda_{1,4}^{X_9^T}(\mathbf{e}_1, \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_7) = -\frac{2}{3}\langle \mathbf{e}_8 \rangle_{1,1}^{X_9^T}.$$

Thus reconstruction of the genus-1 primary correlators follows.

J_{10}^T**-case:**

For $n \geq 2$, we integrate the class $\Lambda_{1,n+2}^{J_{10}^T}(\mathbf{e}_1, \mathbf{e}_{11}, \mathbf{e}_8, \mathbf{e}_4, \mathbf{e}_{10}, \cdots, \mathbf{e}_{10})$ over the Getzler's relation. The non-zero contribution of integrating over $\delta_{2,2}$ comes from three decorated dual graphs. One of them is



Overall, we have

$$\int_{\pi^{-1}_{4,n-2}(\delta_{2,2})} \Lambda^{J^T_{10}}_{1,n+2}(\mathbf{e}_1,\mathbf{e}_{11},\mathbf{e}_8,\mathbf{e}_4,\mathbf{e}_{10},\cdots,\mathbf{e}_{10}) = 3\langle \mathbf{e}_{10},\cdots,\mathbf{e}_{10}\rangle^{J^T_{10}}_{1,n}. \tag{3.12}$$

Then integrations of $\Lambda^{J^T_{10}}_{1,n+2}(\mathbf{e}_1,\mathbf{e}_{11},\mathbf{e}_8,\mathbf{e}_4,\mathbf{e}_{10},\cdots,\mathbf{e}_{10})$ on other strata in Getzler's relation only produce genus-1 correlators with lower insertions and genus-0 correlators. Thus the reconstruction follows for $n \geq 2$.

For $n = 1$, we integrate the class $\Lambda^{J^T_{10}}_{1,4}(\mathbf{e}_8,\mathbf{e}_8,\mathbf{e}_8,\mathbf{e}_{10})$. The unique genus-1 correlators contribution comes from $\delta_{3,4}$. We have

$$\int_{\delta_{3,4}} \Lambda^{J^T_{10}}_{1,4}(\mathbf{e}_8,\mathbf{e}_8,\mathbf{e}_8,\mathbf{e}_{10}) = \frac{4}{3}\langle \mathbf{e}_{10}\rangle^{J^T_{10}}_{1,1}.$$

All the other contributions are of genus-0 correlators. Thus the reconstruction holds. □

$g = 0$

In genus-0, both FJRW theory and Saito theory are well-defined for $\mathbf{t} = 0$. The ancestor correlators can obviously be expressed by ordinary correlators with $\mathbf{t} = 0$.

Proposition III.5. *Using WDVV equations, all genus-0 primary correlators of FJRW theory for the elliptic singularities P_8, X_9^T, J_{10}^T are uniquely determined by the pairing, the 3-point correlators, and $\langle \mathbf{e}_x,\mathbf{e}_x,\mathbf{e}_x,\mathbf{e}_{xyz}\rangle_0^{P_8}$, $\langle \mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z,\mathbf{e}_{xyz}\rangle_0^{P_8}$, $\langle \mathbf{e}_1,\mathbf{e}_5,\mathbf{e}_7,\mathbf{e}_7\rangle_0^{X_9^T}$, $\langle \mathbf{e}_1,\mathbf{e}_8,\mathbf{e}_5,\mathbf{e}_{10}\rangle_0^{J_{10}^T}$, $\langle \mathbf{e}_8,\mathbf{e}_8,\mathbf{e}_8,\mathbf{e}_{10}\rangle_0^{J_{10}^T}$, $\langle \mathbf{e}_1,\mathbf{e}_1,\mathbf{e}_4,\mathbf{e}_{10}\rangle_0^{J_{10}^T}$ respectively.*

We first introduce some useful concepts.

Definition III.6. We call a homogeneous element γ *primitive* if it cannot be written as $\gamma = \gamma_1 \star \gamma_2$ for $\deg_W \gamma_1$ and $\deg_W \gamma_2$ nonzero.

Definition III.7. We call a genus-0 primary correlator a *basic correlator* if there are at most two non-primitive insertions, neither of which are 1.

Our general scheme is following recursion formula from the WDVV equation.

Lemma III.8. *We can reconstruct genus-0 primary correlators of FJRW theory for the three cases by basic correlators with at most four marked points.*

Proof. We recall the WDVV equation for the FJRW invariant,

$$\begin{aligned}\langle\gamma_1,\gamma_2,\delta_S,\gamma_3\star\gamma_4\rangle_{0,n+3} =& I(n)-\langle\gamma_1\star\gamma_2,\delta_S,\gamma_3,\gamma_4\rangle_{0,n+3}\\ &+\langle\gamma_1\star\gamma_3,\delta_S,\gamma_2,\gamma_4\rangle_{0,n+3}+\langle\gamma_1,\gamma_3,\delta_S,\gamma_2\star\gamma_4\rangle_{0,n+3}.\end{aligned}\tag{3.13}$$

where $S=S(n):=\{1,\cdots,n\}, \delta_A:=\alpha_{A_1},\cdots,\alpha_{A_{|A|}}$ and $A=\{A_1,\cdots,A_{|A|}\}$,

$$I(n)=\sum_{\gamma_2\leftrightarrows\gamma_3}\sum_{\substack{A\sqcup B=S\\ A,B\neq\emptyset}}\mathrm{Sign}(\gamma_2,\gamma_3)\langle\gamma_1,\gamma_3,\delta_A,\mu\rangle_{0,|A|+3}\eta^{\mu,\nu}\langle\nu,\delta_B,\gamma_2,\gamma_4\rangle_{0,n+3-|A|}.$$

$\sum_{\gamma_2\leftrightarrows\gamma_3}$ means exchange γ_2,γ_3, and sum up. $\mathrm{Sign}(\gamma_2,\gamma_3)=1, \mathrm{Sign}(\gamma_3,\gamma_2)=-1$. Here we also use the Einstein summation convention for μ,ν. According to [FJR2] Lemma 6.2.6 and Lemma 6.2.8, using the above WDVV equation, all genus-0 primary correlators can be reconstructed uniquely from basic correlators. For all the three singularities listed above, the selection rule (3.5) implies the number of marked points for a basic correlator $\langle\alpha_1,\cdots,\alpha_k\rangle_{0,k}$ should satisfy

$$k-2=\sum_{i=1}^{k}\deg_W\alpha_i\leq(k-2)P+2.$$

where for the singularity W, P is the maximum complex degree for the corresponding FJRW-primitive class. We can easily compute $P=1/3$ for P_8, J_{10}^T, and $P=1/4$ for X_9^T. Thus, for the X_9^T case, $k=4$. For the other two cases, $k=5$. We list all the basic 5-point correlators. Up to symmetry, they are $\langle\mathbf{e}_x,\mathbf{e}_x,\mathbf{e}_x,\mathbf{e}_{xyz},\mathbf{e}_{xyz}\rangle_{0,5}^{P_8}$, $\langle\mathbf{e}_x,\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_{xyz},\mathbf{e}_{xyz}\rangle_{0,5}^{P_8}$, $\langle\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z,\mathbf{e}_{xyz},\mathbf{e}_{xyz}\rangle_{0,5}^{P_8}$ and $\langle\mathbf{e}_8,\mathbf{e}_8,\mathbf{e}_8,\mathbf{e}_{10},\mathbf{e}_{10}\rangle_{0,5}^{J_{10}^T}$.

Now we apply the WDVV equation (3.13) to $\langle\mathbf{e}_x,\mathbf{e}_x,\mathbf{e}_x,\mathbf{e}_{xyz},\mathbf{e}_{xyz}\rangle_{0,5}^{P_8}$. We choose $\gamma_1=\mathbf{e}_x,\gamma_2=\mathbf{e}_{xyz},\gamma_3=\mathbf{e}_{xz},\gamma_4=\mathbf{e}_y,\delta_1=\delta_2=\mathbf{e}_x$, then $\mathbf{e}_x\star\mathbf{e}_{xyz}=\mathbf{e}_x\star\mathbf{e}_{xz}=\mathbf{e}_{xyz}\star\mathbf{e}_y=0$, and the reconstruction follows. The other three cases are reconstructed similarly. □

Now we give the proof of Proposition III.5, *Proof.* We classify all genus-0 4-point *basic* primary correlators.

For P_8 case: We list all the possible non-vanishing basic 4-point correlators up to symmetry. They are:

1. $\langle \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_{xyz}\rangle_0^{P_8}, \langle \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_{xy}, \mathbf{e}_{xz}\rangle_0^{P_8}$;
2. $\langle \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z, \mathbf{e}_{xyz}\rangle_0^{P_8}, \langle \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_{yz}, \mathbf{e}_{yz}\rangle_0^{P_8}, \langle \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_{xz}, \mathbf{e}_{yz}\rangle_0^{P_8}$;
3. $\langle \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_{xyz}\rangle_0^{P_8}, \langle \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_{xz}, \mathbf{e}_{xz}\rangle_0^{P_8}, \langle \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_{xy}, \mathbf{e}_{xz}\rangle_0^{P_8}$;
4. $\langle \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_{xy}, \mathbf{e}_{xy}\rangle_0^{P_8}, \langle \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_{xy}, \mathbf{e}_{xy}\rangle_0^{P_8}$.

Applying the WDVV equation (3.13) over and over again, we can show that all the correlators can be expressed as the scalar multiples of the first one in every row. For example,

$$\langle \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_{xy}, \mathbf{e}_{xz}\rangle_0^{P_8} = \langle \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_x \star \mathbf{e}_y, \mathbf{e}_{xz}\rangle_0^{P_8} = \langle \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_y \star \mathbf{e}_{xz}\rangle_0^{P_8}.$$

Other cases are similar and we leave them to readers as an exercise. Moreover, the scalar is determined by 3-point correlators which are the initial conditions of our reconstruction. Furthermore, we have vanishing results for the last two rows. For example, as $\mathbf{e}_x \star \mathbf{e}_{xy} = \mathbf{e}_x \star \mathbf{e}_x = \mathbf{e}_{xy} \star \mathbf{e}_y = 0$, WDVV equation (3.13) implies

$$\langle \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_{xy}, \mathbf{e}_x \star \mathbf{e}_y\rangle_0^{P_8} = 0.$$

Thus we only need to compute $\langle \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_x, \mathbf{e}_{xyz}\rangle_0^{P_8}$, and $\langle \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z, \mathbf{e}_{xyz}\rangle_0^{P_8}$.

Remark III.9. The above vanishing results can also be obtained by the line bundle criterion (3.6). The same applies to J_{10}^T case for $\langle \mathbf{e}_1, \mathbf{e}_8, \mathbf{e}_5, \mathbf{e}_{10}\rangle_0^{J_{10}^T} = 0$. However, there is no such criterion in the B-model. Here, we stick with the WDVV equation which applies for both the A-model and the B-model.

X_9^T**-case:** There are 18 basic 4-point correlators. Using the WDVV equation,

$$\langle \mathbf{e}_1, \alpha, \beta, \mathbf{e}_5 \star \gamma\rangle_0^{X_9^T} + \langle \mathbf{e}_1 \star \alpha, \beta, \mathbf{e}_5, \gamma\rangle_0^{X_9^T} = \langle \mathbf{e}_1, \gamma, \beta, \mathbf{e}_5 \star \alpha\rangle_0^{X_9^T} + \langle \mathbf{e}_1 \star \gamma, \beta, \mathbf{e}_5, \alpha\rangle_0^{X_9^T}, \tag{3.14}$$

$$(3.15)\quad \langle \mathbf{e}_5, \alpha, \beta, \mathbf{e}_1 \star \xi \rangle_0^{X_9^T} + \langle \mathbf{e}_5 \star \alpha, \beta, \mathbf{e}_1, \xi \rangle_0^{X_9^T} = \langle \mathbf{e}_5, \xi, \beta, \mathbf{e}_1 \star \alpha \rangle_0^{X_9^T} + \langle \mathbf{e}_5 \star \xi, \beta, \mathbf{e}_1, \alpha \rangle_0^{X_9^T}.$$

We can choose as follows:

- $\alpha = \mathbf{e}_2, \mathbf{e}_{10}, x\mathbf{e}_0, \beta = \mathbf{e}_1, \mathbf{e}_5, \gamma = \mathbf{e}_7, \xi = \mathbf{e}_{11}$.
- $\alpha = \mathbf{e}_8, \beta = \mathbf{e}_1$ or $\mathbf{e}_5, \gamma = \mathbf{e}_5, \xi = \mathbf{e}_1$.
- $\alpha = \mathbf{e}_{11}, \beta = \mathbf{e}_1, \gamma = \mathbf{e}_{10}$ in case of (3.14).

There are 17 equations among the 18 basic 4-point correlators. For example, the last choice gives

$$\langle \mathbf{e}_1, \mathbf{e}_{11}, \mathbf{e}_1, \mathbf{e}_{11} \rangle_0^{X_9^T} + \langle \mathbf{e}_8, \mathbf{e}_1, \mathbf{e}_5, \mathbf{e}_{10} \rangle_0^{X_9^T} = \langle \mathbf{e}_7, \mathbf{e}_1, \mathbf{e}_5, \mathbf{e}_{11} \rangle_0^{X_9^T}.$$

By tedious simplification, we find that all the 18 basic 4-point correlators are scalar multiple of $\langle \mathbf{e}_1, \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_7 \rangle_0^{X_9^T}$.

For J_{10}^T-case: We use the same technique. Finally, the basic 4-point correlators are all scalar multiple of the following three special ones: $\langle \mathbf{e}_1, \mathbf{e}_8, \mathbf{e}_5, \mathbf{e}_{10} \rangle_0^{J_{10}^T}$, $\langle \mathbf{e}_8, \mathbf{e}_8, \mathbf{e}_8, \mathbf{e}_{10} \rangle_0^{J_{10}^T}$, $\langle \mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_{10} \rangle_0^{J_{10}^T}$.

□

The 4-point genus-0 FJRW invariants

Let $\Xi(\rho_1, \rho_2, \rho_3)$ be a degree 1 monomial with leading coefficient 1. For simplicity, we denote by $\langle \Xi, \rho_{-1} \rangle_{0,4}^{W^T}$ a basic correlator such that the first three insertions give a factorization of Ξ. For example, let $\Xi(\rho_1, \rho_2, \rho_3) = \rho_1^2 \rho_2^2$; then the notation $\langle \Xi, \rho_{-1} \rangle_{0,4}^{W^T}$ represents any of the following choices of correlators:

$$\langle \rho_1, \rho_1, \rho_2^2, \rho_{-1} \rangle_{0,4}^{W^T}, \langle \rho_1, \rho_2, \rho_1\rho_2, \rho_{-1} \rangle_{0,4}^{W^T}, \langle \rho_2, \rho_2, \rho_1^2, \rho_{-1} \rangle_{0,4}^{W^T}.$$

The WDVV equations guarantee that $\langle \Xi, \rho_{-1} \rangle_{0,4}^{W^T}$ does not depend on the choices of the factorization.

Lemma III.10. *Let W^T be an ISES; then all FJRW correlators for (W^T, G_{W^T}) can be reconstructed from the FJRW algebra, and the basic 4-point FJRW correlators $\langle \Xi, \rho_{-1}\rangle_{0,4}^{W^T}$. Furthermore, if W^T is an ISES as in Tables 3.1, 3.2, or 3.3, then*

$$\langle \Xi(\rho_1, \rho_2, \rho_3), \rho_{-1}\rangle_{0,4}^{W^T} = \begin{cases} q_i^T & \text{if } \Xi = M_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

where M_i are the homogeneous monomials such that $W = M_1 + M_2 + M_3$.

Proof. We already see for three special simple elliptic singularities $W^T = X_1^3 + X_2^3 + X_3^3$, $X_1^3 + X_1X_2^2 + X_2X_3^2$ and $X_1^3 + X_1X_2^2 + X_3^3$, their FJRW correlators with symmetry group G_{W^T} can be reconstructed from their FJRW algebra and some basic 4-point correlators. We apply the same method to all cases of simple elliptic singularities here. Finally, using WDVV equations in each case, it is again not hard to verify all 4-point basic correlators without insertion ρ_{-1} can be reconstructed too.

For the second part of the lemma, we use WDVV and the concavity to compute FJRW correlators. We show that the argument works for singularities of Fermat type and of loop type. Other cases are similar. For a Fermat type singularity, put $M_i = X_i^{1/q_i^T}$, since all insertions are narrow, we apply the Concavity Axiom (3.9) to compute

$$\langle \rho_i, \rho_i, \rho_i^{1/q_i^T - 2}, \rho_{-1}\rangle_{0,4}^{W^T}. \tag{3.17}$$

Note that $\deg \mathscr{L}_i = -2$ and the degree shifting numbers are $(2q_i^T, 2q_i^T, 1 - q_i^T, 1 - q_i^T)$, thus the dual graphs will have $\Theta_\Gamma = 0, 0, 1 - 3q_i^T$. The correlator (3.17) becomes

$$\frac{1}{2}\left(B_2(q_i^T) + B_2(1 - 3q_i^T) + 2B_2(0) - 2B_2(q_i^T) - 2B_2(1 - q_i^T)\right) = q_i^T.$$

For loop type, $W^T = X_1^2X_3 + X_1X_2^2 + X_2X_3^2$. Let us compute $\langle \rho_1, \rho_1, \rho_2, \rho_{-1}\rangle_{0,4}^{W^T}$, which is not concave. However, the Concavity Axiom (3.9) implies

$$\langle \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_7, \mathbf{e}_2\rangle_{0,4}^{W^T} = -\frac{2}{9}.$$

On the other hand, WDVV equations show

$$\begin{cases} \langle \mathbf{e}_1 \bullet \mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_7, \mathbf{e}_2 \rangle_{0,4}^{W^T} + \langle \mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_7 \bullet \mathbf{e}_2 \rangle_{0,4}^{W^T} = \langle \mathbf{e}_7 \bullet \mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_2 \rangle_{0,4}^{W^T}; \\ \\ \langle \mathbf{e}_4 \bullet \mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_7, \mathbf{e}_2 \rangle_{0,4}^{W^T} + \langle \mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_7 \bullet \mathbf{e}_2 \rangle_{0,4}^{W^T} = \langle \mathbf{e}_4 \bullet \mathbf{e}_7, \mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_2 \rangle_{0,4}^{W^T}; \end{cases}$$

We observe up to symmetry, $\langle \mathbf{e}_5, \mathbf{e}_1, \mathbf{e}_7, \mathbf{e}_2 \rangle_{0,4}^{W^T} = \langle \mathbf{e}_8, \mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_2 \rangle_{0,4}^{W^T}$. Recall the ring relations in Example III.1, we obtain

$$\langle \rho_1, \rho_1, \rho_2, \rho_{-1} \rangle_{0,4}^{W^T} = \langle \mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_6 \rangle_{0,4}^{W^T} = \frac{1}{3}. \quad \square$$

3.4 Classification of good FJRW theories

Conjecture III.11. *For W, W' be invertible simple elliptic singularities and $G_W, G_{W'}$ be theis maximal diagonal symmetry groups, then the FJRW theory of (W, G_W) is equivalent to the FJRW theory of $(W', G_{W'})$ if and only if*

$$\mu_{W^T} = \mu_{W'^T}, \quad j_{W^T}(0) = j_{W'^T}(0).$$

In this section, we will classify the FJRW theory of those good cases and prove the conjecture is true among thoses cases. For other cases, due to the [Kr], the conjecture is true in ring structure level.

Lemma III.12. *If W_1 and W_2 belong to the same equivalence class, then there exists a linear isomorphism $\Psi : \mathscr{H}_{W_1^T} \to \mathscr{H}_{W_2^T}$ that induces an isomorphism of the FJRW rings and the corresponding 4-point basic correlators with a degree 1 insertion. According to the reconstruction lemma III.2, all higher genus FJRW correlators of (W_1, G_{W_1}) and (W_2, G_{W_2}) are identified under this ring isomorphism.*

Proof. In order to prove the lemma we will construct explicitly a linear isomorphisms Ψ inducing the ring isomorphisms; then one has to check that they also preserve the corresponding 4-point basic correlators with a degree 1 insertion.

For simplicity, we add a superscript on the ring generators for each singularity. The superscripts f, c or l mean that the singularity is respectively of Fermat, chain, or loop type. A superscript with two entries means it is a sum of two different types. Let us take the first and the last FJRW rings in the case of $(\mu, j) = (8, 0)$ as an example. We choose a linear transformation,

$$\Psi\begin{pmatrix} \rho_1^f/\lambda_1 \\ \\ \rho_2^f/\lambda_2 \\ \\ \rho_3^f/\lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \\ 1 & \mathbf{e}[\frac{1}{3}] & \mathbf{e}[\frac{2}{3}] \\ \\ 1 & \mathbf{e}[\frac{2}{3}] & \mathbf{e}[\frac{1}{3}] \end{pmatrix}\begin{pmatrix} \rho_1^{lf} \\ \\ \rho_2^{lf} \\ \\ \rho_3^{lf} \end{pmatrix}.$$

The parameters $\lambda_1, \lambda_2, \lambda_3$ could be chosen to be arbitrary non-zero complex numbers. The relations in the FJRW rings do not depend on the choice. However, we have to choose λ_i in such a way that the basic 4-point correlators agree, i.e.,

$$\lambda_1^4\lambda_2\lambda_3 = \mathbf{e}[\frac{1}{3}]\,\lambda_1\lambda_2^4\lambda_3 = \mathbf{e}[\frac{2}{3}]\,\lambda_1\lambda_2\lambda_3^4 = -\frac{1}{81}. \tag{3.18}$$

We notice that

$$\rho_{-1}^f = \rho_1^f \bullet \rho_2^f \bullet \rho_3^f, \quad \rho_{-1}^l = \rho_1^l \bullet \rho_2^l \bullet \rho_3^l.$$

Now one can check that

$$\langle \Psi(\rho_1^f), \Psi(\rho_1^f), \Psi(\rho_1^f), \Psi(\rho_{-1}^f)\rangle_{0,4} = -81\lambda_1^4\lambda_2\lambda_3\langle \rho_1^l, \rho_1^l, \rho_1^l, \rho_{-1}^l\rangle_{0,4} = \frac{1}{3}.$$

We check all the nonzero 4-point genus-0 correlators listed in Lemma III.10 and obtain the other two identities in (3.18).

Another example between Fermat $E_6^{(1,1)}$ and loop $E_6^{(1,1)}$

Let us give an isomorphism between FJRW theory of $(X_1^3+X_2^3+X_3^3, G_{W^T})$ and FJRW theory of $(X_1^2X_3+X_1X_2^2+X_2X_3^2, G_{W^T})$,

$$-3\rho_{-1}\mapsto\rho_{-1};\qquad \begin{pmatrix} c_1\rho_1 \\ c_2\rho_2 \\ c_3\rho_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 1 & \mathbf{e}[\frac{1}{3}] & \mathbf{e}[\frac{2}{3}] \\ 1 & \mathbf{e}[\frac{2}{3}] & \mathbf{e}[\frac{1}{3}] \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}$$

where constants c_i satisfy

$$c_1^3=\mathbf{e}[\frac{1}{3}]c_2^3=\mathbf{e}[\frac{2}{3}]c_3^3=-9.$$

Other examples

The other cases are similar. Let us list the corresponding linear transformations which preserve FJRW correlators of all genera. We are only interested in those singularities listed in Table 3.1, 3.2, 3.3.

- $(\mu, j)=(9,0)$,

$$\Psi\begin{pmatrix} \rho_1^c \\ \rho_2^c \end{pmatrix} = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \begin{pmatrix} \rho_1^{cf} \\ \rho_2^{cf} \end{pmatrix}. \quad -2\lambda_1^4\lambda_2^4=8\lambda_1\lambda_2^7=1.$$

- $(\mu, j)=(9,1)$,

$$\Psi\begin{pmatrix} \rho_1^f \\ \rho_2^f \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_1 \\ -\lambda_2 & \lambda_2 \end{pmatrix} \begin{pmatrix} \rho_1^{lf} \\ \rho_2^{lf} \end{pmatrix}, \quad -64\lambda_1^6\lambda_2^2=64\lambda_1^2\lambda_2^6=1.$$

- $(\mu, j) = (10, 0)$,

$$\Psi \begin{pmatrix} \rho_1^f \\ \\ \rho_2^f \end{pmatrix} = \begin{pmatrix} \lambda_1 & \\ & \\ & \lambda_2 \end{pmatrix} \begin{pmatrix} \rho_1^{cf} \\ \\ \rho_2^{cf} \end{pmatrix}, \quad 8\lambda_1^{10}\lambda_2 = -2\lambda_1^4\lambda_2^4 = 1. \quad \square$$

3.5 Convergence

We will prove the following statement in Chapter IX.

Lemma III.13. *The FJRW ancestor correlator* $\langle\langle\tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n)\rangle\rangle_{g,n}^{W,G}(s\rho_{-1})$ *is convergent at* $s = 0$*.*

Now we use it to prove the convergence of the FJRW part in Theorem I.4.

Proof. We can assume $\mathbf{t} = s\rho_{-1} + \sum_{i\geq 0} t_i\phi_i$, where ϕ_i ranges over all elements in the basis except ρ_{-1}. By Selection rule (3.5), $\langle\langle\tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n)\rangle\rangle_{g,n}^{W,G}(\mathbf{t})$ can be written as a polynomial of t_i, with each coefficient some FJRW ancestor function valued at $s\rho_{-1}$. The degree of each monomial just depends on g and n. Now the convergence is an easy consequence from the previous lemma. $\square$

CHAPTER IV

Global B-model for simple elliptic singularities

4.1 Saito's theory

Let W be an invertible polynomial from Table 1.1. We would like to recall Saito's theory of primitive forms which yields a Frobenius structure on the miniversal deformation space $\mathcal{M}$. Following Givental's higher genus reconstruction formalism we will introduce the total ancestor potential of W. Finally, we will derive a system of hypergeometric equations that determines the restriction of the flat coordinates of the Frobenius manifold $\mathcal{M}$ to Σ.

Miniversal deformation

Let

$$\mathscr{Q}_W = \mathbb{C}[X_1, X_2, X_3]/(\partial_{X_1} W, \partial_{X_2} W, \partial_{X_3} W).$$

be the *Jacobian algebra* or *local algebra* of W. Let us fix a set $\mathfrak{R}$ of weighted homogeneous monomials

$$\phi_{\mathbf{r}}(\mathbf{X}) = X_1^{r_1} X_2^{r_2} X_3^{r_3}, \quad \mathbf{r} = (r_1, r_2, r_3), \tag{4.1}$$

such that their projections in $\mathscr{Q}_W$ form a basis. The dimension of the Jacobi algebra, i.e., the number of the above monomials, is called the *multiplicity* of the critical point or the *Milnor number* and it will be denoted by μ. There is precisely one monomial of top degree, say $\phi_{\mathbf{m}}$, $\mathbf{m} = (m_1, m_2, m_3)$. We fix a deformation of W of the following form:

$$W_\sigma(\mathbf{X}) = W(\mathbf{X}) + \sigma\,\phi_{\mathbf{m}}(\mathbf{X}), \quad \sigma \in \Sigma, \tag{4.2}$$

where $\Sigma \subset \mathbb{C}$ is the set of all $\sigma \in \mathbb{C}$ such that $W_\sigma(\mathbf{X})$ has only isolated critical points. Such deformations do not change the multiplicity of the critical point at $\mathbf{X} = 0$. The polynomials (4.2) correspond to families of simple elliptic singularities of type $E^{(1,1)}_{\mu-2}$ (see [Sa2]). More generally, a miniversal deformation (see e.g. [ArGV]) of W can be constructed in the form

$$F(\mathbf{s}, \mathbf{X}) = W(\mathbf{X}) + \sum_{\mathbf{r}\in\Re} s_{\mathbf{r}}\,\phi_{\mathbf{r}}(\mathbf{X}). \tag{4.3}$$

It is convenient to adopt two notations for the deformation parameters. Namely, put

$$\mathbf{s} = \{s_{\mathbf{r}}\}_{\mathbf{r}\in\Re} = (s_{-1}, s_0, s_1, \ldots, s_{\mu-2}),$$

where the second equality is obtained by putting an order on the elements $\mathbf{r} \in \Re$ and enumerating them with the integers from -1 to $\mu - 2$ in such a way that

$$s_{-1} = s_{\mathbf{m}} = \sigma, \quad s_0 = s_{\mathbf{0}}, \quad \mathbf{0} = (0, 0, 0) \in \Re.$$

The space of miniversal deformations, i.e., the range of the parameters $s_{\mathbf{r}}$ is then defined to be the affine space $\mathcal{M} = \Sigma \times \mathbb{C}^{\mu-1}$. Furthermore, each $s_{\mathbf{r}}$ is assigned a degree so that F is weighted-homogeneous of degree 1. Note that the parameter $s_{\mathbf{m}} = \sigma$ has degree 0. Following the terminology in physics, we call $s_{\mathbf{m}}$ and $\phi_{\mathbf{m}}$ *marginal*. Note that $W_\sigma(\mathbf{X})$ is the restriction of $F(\mathbf{s}, \mathbf{X})$ to the subspace Σ of marginal deformations. Except for W of Fermat type, there is more than one choice of a marginal monomial. For example, both $X_1X_2X_3$ and $X_1^4X_3$ are marginal for $W = X_1^3X_2 + X_2^2 + X_3^3$.

Multiplication

Let C be the critical variety of the miniversal deformation F (see (4.3)), i.e., the support of the sheaf

$$\mathcal{O}_C := \mathcal{O}_X / \langle \partial_{X_1}F, \partial_{X_2}F, \partial_{X_3}F \rangle,$$

where $X = \mathcal{M} \times \mathbb{C}^3$. Let $q : X \to \mathcal{M}$ be the projection on the first factor. The Kodaira–Spencer map ($\mathcal{T}_\mathcal{M}$ is the sheaf of holomorphic vector fields on $\mathcal{M}$)

$$\mathcal{T}_\mathcal{M} \longrightarrow q_* \mathcal{O}_C, \quad \partial/\partial s_i \mapsto \partial F/\partial s_i \bmod (F_{X_1}, F_{X_2}, F_{X_3})$$

is an isomorphism, which implies that for any $\mathbf{s} \in \mathcal{M}$, the tangent space $T_\mathbf{s}\mathcal{M}$ is equipped with an associative commutative multiplication $\bullet_\mathbf{s}$ depending holomorphically on $\mathbf{s} \in \mathcal{M}$. If in addition we have a volume form $\omega = g(\mathbf{s}, \mathbf{x})d^3\mathbf{x}$, where $d^3\mathbf{x} = dX_1 \wedge dX_2 \wedge dX_3$ is the standard volume form, then $q_*\mathcal{O}_C$ (hence $\mathcal{T}_\mathcal{M}$ as well) is equipped with the *residue pairing*:

$$\langle \psi_1, \psi_2 \rangle = \frac{1}{(2\pi i)^3} \int_{\Gamma_\epsilon} \frac{\psi_1(\mathbf{s}, \mathbf{y})\psi_2(\mathbf{s}, \mathbf{y})}{F_{y_1} F_{y_2} F_{y_3}}\, \omega, \tag{4.4}$$

where $\mathbf{y} = (y_1, y_2, y_3)$ is a unimodular coordinate system for the volume form, i.e., $\omega = d^3\mathbf{y}$, and Γ_ϵ is a real 3-dimensional cycle supported on $|F_{X_i}| = \epsilon$ for $1 \le i \le 3$.

Given a semi-infinite cycle

$$\mathcal{A} \in \varprojlim H_3(\mathbb{C}^3, (\mathbb{C}^3)_{-m}; \mathbb{C}) \cong \mathbb{C}^\mu, \tag{4.5}$$

where

$$(\mathbb{C}^3)_m = \{\mathbf{x} \in \mathbb{C}^3 \mid \mathrm{Re}(F(\mathbf{s}, \mathbf{x})/z) \le m\}. \tag{4.6}$$

Put

$$J_\mathcal{A}(\mathbf{s}, z) = (-2\pi z)^{-3/2}\, z d_\mathcal{M} \int_\mathcal{A} e^{F(\mathbf{s},\mathbf{x})/z} \omega, \tag{4.7}$$

where $d_\mathcal{M}$ is the de Rham differential on $\mathcal{M}$. The oscillatory integrals $J_\mathcal{A}$ are, by definition, sections of the cotangent sheaf $\mathcal{T}^*_\mathcal{M}$.

According to Saito's theory of primitive forms [Sa1], there exists a volume form ω such that the residue pairing is flat and the oscillatory integrals satisfy a system of differential equations, which in flat-homogeneous coordinates $\mathbf{t} = (t_{-1}, t_0, \ldots, t_{\mu-2})$ have the form

$$z\partial_i J_\mathcal{A}(\mathbf{t}, z) = \partial_i \bullet_\mathbf{t} J_\mathcal{A}(\mathbf{t}, z), \tag{4.8}$$

where $\partial_i := \partial/\partial t_i$ $(-1 \leq i \leq \mu - 2)$ and the multiplication is defined by identifying vectors and covectors via the residue pairing. Using the residue pairing, the flat structure, and the Kodaira–Spencer isomorphism we have the following isomorphisms:

$$T^*\mathcal{M} \cong T\mathcal{M} \cong \mathcal{M} \times T_0\mathcal{M} \cong \mathcal{M} \times \mathcal{Q}_W.$$

Due to the homogeneity, the integrals satisfy a differential equation with respect to the parameter $z \in \mathbb{C}^*$:

$$(z\partial_z + E)J_{\mathcal{A}}(\mathbf{t}, z) = \Theta\, J_{\mathcal{A}}(\mathbf{t}, z), \tag{4.9}$$

where

$$E = \sum_{i=-1}^{\mu-2} d_i t_i \partial_i, \quad (d_i := \deg t_i = \deg s_i),$$

is the *Euler vector field* and Θ is the so-called *Hodge grading operator*

$$\Theta : \mathcal{T}^*_{\mathcal{M}} \to \mathcal{T}^*_{\mathcal{M}}, \quad \Theta(dt_i) = \left(\frac{1}{2} - d_i\right) dt_i.$$

The compatibility of the system (4.8)–(4.9) implies that the residue pairing, the multiplication, and the Euler vector field give rise to a *conformal Frobenius structure* of conformal dimension 1. We refer to B. Dubrovin [Du] for the definition and more details on Frobenius structures and to C. Hertling [He] or to Atsushi–Saito [ST] for more details on constructing a Frobenius structure from a primitive form.

4.2 Primitive forms and global B-model

The classification of primitive forms in general is a very difficult problem. In the case of simple elliptic singularities however, all primitive forms are known (see [Sa1]). They are given by $\omega = d^3\mathbf{x}/\pi_A(\sigma)$, where $\pi_A(\sigma)$ is the period (4.11). As we will prove below, these periods are solutions to the hypergeometric equation (4.14), so a primitive form may be equivalently fixed by fixing a solution to the differential equation that does not vanish on Σ. Note that since $\pi_A(\sigma)$ is multi-valued function, the corresponding Frobenius structure on $\mathcal{M}$ is multi-valued as well. In other words, the primitive form gives rise to a Frobenius structure on the universal cover $\widetilde{\mathcal{M}} \cong \mathbb{H} \times \mathbb{C}^{\mu-1}$.

The key to the primitive form is the Picard-Fuchs differential equation for the periods of the so-called *elliptic curve at infinity*

$$E_\sigma := \Big\{[X_1 : X_2 : X_3] \in \mathbb{CP}^2(c_1, c_2, c_3) \,\Big|\, W_\sigma = 0\Big\}, \tag{4.10}$$

where $c_i = d/o_i$, $1 \le i \le 3$ and d is the least common multiple of o_1, o_2, and o_3. Note that E_σ are the fibers of an elliptic fibration over $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ whose non-singular fibers are parametrized by $\Sigma \subset \mathbb{C} \subset \mathbb{CP}^1$. Note that $\mathrm{Res}_{E_\sigma}\Omega$, where

$$\Omega := \frac{dX_1 \wedge dX_2 \wedge dX_3}{dW_\sigma},$$

is a Calabi-Yau form of the elliptic curve E_σ. For every $A \in H_1(E_\sigma)$, we define the period integral

$$\pi_A(\sigma) = \int_A \mathrm{Res}_{E_\sigma}\Omega. \tag{4.11}$$

It is well known that the period integrals are solutions to a Fuchsian differential equation. For our purposes, since we have to deal with many examples, it is convenient to follow the approach of S. Gährs (see [Ga]. We define a *charge vector* $\vec{L}$, where

$$\vec{L} = (l_1, l_2, l_3, -l) \in \mathbb{Z}^4$$

by choosing the minimal $l \in \mathbb{Z}_{>0}$ such that

$$(l_1, l_2, l_3) = l\,\mathbf{m}\,E_W^{-1}, \quad \mathbf{m} = (m_1, m_2, m_3). \tag{4.12}$$

We define a *charge constant* C, where

$$C = \prod_{i=1}^{3} \left(-\frac{l_i}{l}\right)^{l_i}. \tag{4.13}$$

Lemma IV.1. *The Picard-Fuchs equation for the elliptic periods is*

$$\delta(\delta-1)\,\pi_A = C\,\sigma^l(\delta+l\alpha)(\delta+l\beta)\,\pi_A, \tag{4.14}$$

where $\delta = \sigma\partial/\partial\sigma$ and α and β are some rational numbers, such that $\alpha+\beta = 1-\frac{1}{l}$.

If we put $x = C\,\sigma^l$, $\gamma = \alpha+\beta$, then the above differential equation turns into the standard hypergeometric equation

$$x(1-x)\frac{d^2\pi_A}{dx^2} + (\gamma-(1+\alpha+\beta)x)\,\frac{d\pi_A}{dx} - \alpha\beta\,\pi_A = 0 \tag{4.15}$$

We call (α,β,γ) the *weights* of the Picard-Fuchs equation. They are listed in Table 4.1 for $E_6^{(1,1)}$, Table 4.2 and 4.3 for the other two cases. In particular, the singularities of the Picard–Fuchs equation are at the points

$$p_i = C^{-1/l}\eta^i, \quad 1\le i\le l, \quad \eta = \exp\left(\frac{2\pi\sqrt{-1}}{l}\right), \tag{4.16}$$

and we have $\Sigma = \mathbb{C}\backslash\{p_1,\ldots,p_l\}$.

Table 4.1: $E_6^{(1,1)}$

W	m_1,m_2,m_3	l_1,l_2,l_3	q_1^T,q_2^T,q_3^T	α,β,γ
$X_1^3+X_2^3+X_3^3$	$(1,1,1)$	$1,1,1,3$	$\frac{1}{3},\frac{1}{3},\frac{1}{3}$	$\frac{1}{3},\frac{1}{3},\frac{2}{3}$
$X_1^2X_2+X_2^3+X_3^3$	$(2,0,1)$	$3,-1,1,3$	$\frac{1}{2},\frac{1}{6},\frac{1}{3}$	$\frac{1}{6},\frac{1}{2},\frac{2}{3}$
$X_1^2X_2+X_2^3+X_3^3$	$(0,2,1)$	$0,2,1,3$	$\frac{1}{2},\frac{1}{6},\frac{1}{3}$	$\frac{1}{12},\frac{7}{12},\frac{2}{3}$
$X_1^2X_2+X_1X_2^2+X_3^3$	$(1,1,1)$	$1,1,1,3$	$\frac{1}{3},\frac{1}{3},\frac{1}{3}$	$\frac{1}{3},\frac{1}{3},\frac{2}{3}$
$X_1^2X_2+X_1X_2^2+X_3^3$	$(2,0,1)$	$4,-2,1,,3$	$\frac{1}{3},\frac{1}{3},\frac{1}{3}$	$\frac{1}{12},\frac{7}{12},\frac{2}{3}$
$X_1^2X_2+X_2^2X_3+X_3^3$	$(2,0,1)$	$2,-1,1,2$	$\frac{1}{2},\frac{1}{4},\frac{1}{4}$	$\frac{1}{4},\frac{1}{4},\frac{1}{2}$
$X_1^2X_2+X_2^2X_3+X_3^3$	$(0,3,0)$	$0,3,-1,2$	$\frac{1}{2},\frac{1}{4},\frac{1}{4}$	$\frac{1}{12},\frac{5}{12},\frac{1}{2}$
$X_1^2X_2+X_2^2X_3+X_3^3$	$(0,1,2)$	$0,1,1,2$	$\frac{1}{2},\frac{1}{4},\frac{1}{4}$	$\frac{1}{4},\frac{1}{4},\frac{1}{2}$
$X_1^2X_2+X_2^2X_3+X_1X_3^2$	$(1,1,1)$	$1,1,1,3$	$\frac{1}{3},\frac{1}{3},\frac{1}{3}$	$\frac{1}{3},\frac{1}{3},\frac{2}{3}$
$X_1^2X_2+X_2^2X_3+X_1X_3^2$	$(3,0,0)$	$4,-2,1,3$	$\frac{1}{3},\frac{1}{3},\frac{1}{3}$	$\frac{1}{12},\frac{7}{12},\frac{2}{3}$

Table 4.2: $E_7^{(1,1)}$

W	m_1, m_2, m_3	l_1, l_2, l_3, l	q_1^T, q_2^T, q_3^T	α, β, γ
$X_1^4 + X_2^4 + X_3^2$	$(2,2,0)$	$1,1,0,2$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
$X_1^3 X_2 + X_2^4 + X_3^2$	$(4,0,0)$	$4,-1,0,3$	$\frac{1}{3}, \frac{1}{2}, \frac{1}{6}$	$\frac{1}{12}, \frac{7}{12}, \frac{2}{3}$
$X_1^3 X_2 + X_2^4 + X_3^2$	$(1,3,0)$	$1,2,0,3$	$\frac{1}{3}, \frac{1}{2}, \frac{1}{6}$	$\frac{1}{12}, \frac{7}{12}, \frac{2}{3}$
$X_1^2 X_2 + X_2^2 + X_3^4$	$(2,0,2)$	$2,-1,1,2$	$\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
$X_1^2 X_2 + X_2^2 + X_3^4$	$(0,1,2)$	$0,1,1,2$	$\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
$X_1^3 X_2 + X_1 X_2^3 + X_3^2$	$(4,0,0)$	$3,-1,0,2$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	$\frac{1}{12}, \frac{5}{12}, \frac{1}{2}$
$X_1^3 X_2 + X_1 X_2^3 + X_3^2$	$(2,2,0)$	$1,1,0,2$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
$X_1^3 X_2 + X_2^2 X_3 + X_3^2$	$(1,1,1)$	$1,1,1,3$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}$
$X_1^3 X_2 + X_2^2 X_3 + X_3^2$	$(1,3,0)$	$1,4,-2,3$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{12}, \frac{7}{12}, \frac{2}{3}$
$X_1^3 X_2 + X_2^2 X_3 + X_3^2$	$(4,0,0)$	$4,-2,1,3$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{12}, \frac{7}{12}, \frac{2}{3}$

Table 4.3: $E_8^{(1,1)}$

W	m_1, m_2, m_3	l_1, l_2, l_3, l	q_1^T, q_2^T, q_3^T	α, β, γ	$j(\sigma)/1728$
$X_1^6 + X_2^3 + X_3^2$	$(4,0,1)$	$1,2,0,3$	$\frac{1}{6}, \frac{1}{3}, \frac{1}{2}$	$\frac{1}{12}, \frac{7}{12}, \frac{2}{3}$	$\frac{4\sigma^3}{4\sigma^3+27}$
$X_1^3 X_2 + X_2^2 + X_3^3$	$(1,1,1)$	$1,1,1,3$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}$	$\frac{\sigma^3(\sigma^3-24)^3}{1728(\sigma^3-27)}$
$X_1^3 X_2 + X_2^2 + X_3^3$	$(4,0,1)$	$4,-2,1,3$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{12}, \frac{7}{12}, \frac{2}{3}$	$\frac{64\sigma^3}{64\sigma^3+27}$
$X_1^4 X_2 + X_2^3 + X_3^2$	$(2,2,0)$	$1,1,0,2$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	$\frac{4(3-\sigma^2)^3}{27(4-\sigma^2)}$
$X_1^4 X_2 + X_2^3 + X_3^2$	$(6,0,0)$	$3,-1,0,2$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	$\frac{1}{12}, \frac{5}{12}, \frac{1}{2}$	$\frac{4}{27\sigma^2+4}$

The j-invariant of E_σ is determined by the charge vector. A table of j-invariant is shown here. For some of the computations, we refer to [Co].

Table 4.4: j-invariant

W	$\mathbf{m}$	$\vec{a}$	$j(\sigma)$
$X_1^6 + X_2^3 + X_3^2$	$(4,0,1)$	$(0,0,0,\sigma,-1)$	$\frac{4\sigma^3}{4\sigma^3+27}$
$X_1^3 X_2 + X_2^2 + X_3^3$	$(1,1,1)$	$(\sigma,0,1,0,0)$	$\frac{\sigma^3(\sigma^3-24)^3}{1728(\sigma^3-27)}$
$X_1^3 X_2 + X_2^2 + X_3^3$	$(4,0,1)$	$(0,0,1,\sigma,0)$	$\frac{64\sigma^3}{64\sigma^3+27}$
$X_1^4 X_2 + X_2^3 + X_3^2$	$(2,2,0)$	$(0,-\sigma,0,1,0)$	$\frac{4(3-\sigma^2)^3}{27(4-\sigma^2)}$
$X_1^4 X_2 + X_2^3 + X_3^2$	$(6,0,0)$	$(0,0,0,1,-\sigma)$	$\frac{4}{27\sigma^2+4}$
$X_1^2 X_2 + X_2^2 X_3 + X_3^3$	$(2,0,1)$	$(0,\sigma,0,1,\sigma)$	$\frac{(3-\sigma^2)^3}{27(\sigma^2+1)^2}$
$X_1^2 X_2 + X_2^3 + X_3^3$	$(2,0,1)$	$\left(0,3\sigma^2,0,-3\sigma(1-\sigma^3),(1-\sigma^3)^2\right)$	$\frac{-4\sigma^3}{(1-\sigma^3)^2}$

4.3 Picard-Fuchs systems

Let us denote by

$$X_{\mathbf{s}} = \{\mathbf{X} \in \mathbb{C}^3 \mid F(\mathbf{s}, \mathbf{X}) = 1\}, \quad \mathbf{s} \in \mathcal{M}.$$

The points $\mathbf{s}$ for which $X_{\mathbf{s}}$ is singular form an analytic hypersurface in $\mathcal{M}$ called the *discriminant hypersurface*. Its complement in $\mathcal{M}$ will be denoted by $\mathcal{M}'$. We will be interested in the period integrals

$$\Phi_{\mathbf{r}}(\mathbf{s}) = \int \phi_{\mathbf{r}}(x) \frac{d^3\mathbf{X}}{dF}, \quad \phi_{\mathbf{r}}(\mathbf{X}) = X_1^{r_1} X_2^{r_2} X_3^{r_3}, \quad \mathbf{r} = (r_1, r_2, r_3).$$

They are sections of $\mathscr{H}^{\mathrm{mid}}$, the *cohomology Milnor fibration* on $\mathcal{M}'$ with fibers $H^2(X_{\mathbf{s}}, \mathbb{C})$. Slightly abusing the notation, we denote the restriction to $s_{-1} = \sigma$, $s_i = 0 (0 \le i \le \mu - 2)$ by $\Phi_{\mathbf{r}}(\sigma)$. Following the idea of [Ga], we first obtain a GKZ (Gelfand–Kapranov–Zelevinsky) system of differential equations for the periods. Using that the period integrals are not polynomial in σ (they have singularities at the punctures of Σ) we can reduce the GKZ system to a Picard-Fuchs equation.

The GKZ system

In order to derive the GKZ system, we slightly modify the polynomial W. By definition $W(\mathbf{X}) = \sum_{i=1}^3 \phi_{\mathbf{a}_i}(\mathbf{X})$, where $\mathbf{a}_i$ are the rows of the matrix E_W. Put

$$W_{\mathbf{v},\sigma}(\mathbf{X}) = \sum_{i=1}^{3} v_i \, \phi_{\mathbf{a}_i}(\mathbf{X}) + \sigma \, \phi_{-1}(\mathbf{X}),$$

where $\mathbf{v} = (v_1, v_2, v_3)$ are some complex parameters. For simplicity, we omit $\mathbf{v}$ in the notation if $\mathbf{v} = (1, 1, 1)$. Let us write $X_\sigma^{\mathbf{v},\lambda} = \{\mathbf{X} \in \mathbb{C}^3 \mid W_{\mathbf{v},\sigma}(\mathbf{X}) = \lambda\}$. Then we define the period integrals

$$\Phi_{\mathbf{r}}^{\mathbf{v},\lambda}(\sigma) = \int \phi_{\mathbf{r}}(\mathbf{X}) \frac{d^3\mathbf{X}}{dW_{\mathbf{v},\sigma}}; \tag{4.17}$$

again one should think that the above integral is a section of the vanishing cohomology for $W_{\mathbf{v},\sigma}(\mathbf{X})$. The vanishing cohomology bundle is equipped with a Gauss–Manin connection ∇. The following formulas are well known (see e.g. [ArGV])

$$\begin{aligned}
\nabla_{\partial/\partial\lambda} \int \theta &= \int \frac{d\theta}{dW_{\mathbf{v},\sigma}} \\
\nabla_{\partial/\partial v_i} \int \theta &= -\int \frac{\partial W_{\mathbf{v},\sigma}}{\partial v_i} \frac{d\theta}{dW_{\mathbf{v},\sigma}} + \int \mathrm{Lie}_{\partial/\partial v_i}\theta,
\end{aligned} \tag{4.18}$$

where θ is a 2-form on $\mathbb{C}^3$ possibly depending on the parameters $\mathbf{v}$. Finally, note that rescaling $X_i \mapsto \lambda^{q_i} X_i (1 \le i \le 3)$ yields

$$\Phi_{\mathbf{r}}^{\mathbf{v},\lambda}(\sigma) = \lambda^{\deg \phi_{\mathbf{r}}}\, \Phi_{\mathbf{r}}^{\mathbf{v},1}(\sigma).$$

Let $\delta_i = v_i \partial/\partial v_i (1 \le i \le 3)$ and $\delta = \sigma\partial/\partial\sigma$.

Lemma IV.2. *The period integral $\Phi_{\mathbf{r}}^{\mathbf{v},\lambda}$ satisfies the following system of differential equations:*

$$\begin{aligned}
\partial_\sigma^l \prod_{i:l_i<0} \partial_{v_i}^{-l_i}\, \Phi &= \prod_{i:l_i>0} \partial_{v_i}^{l_i}\, \Phi; \\
(\delta_1, \delta_2, \delta_3)\, E_W\, \Phi + (m_1, m_2, m_3)\delta\, \Phi &= -(1 + r_1, 1 + r_2, 1 + r_3)\Phi.
\end{aligned}$$

where the range for i and j in the first equation on the LHS and the RHS is $0 \le i \le 3$.

Proof. Using (4.18) we get the following differential equations:

$$\partial_{v_i}\, \Phi_{\mathbf{r}}^{\mathbf{v},\lambda} = -\partial_\lambda\, \Phi_{\mathbf{r}+\mathbf{a}_i}^{\mathbf{v},\lambda}, \quad 1 \le i \le 3,$$

and

$$\partial_\sigma\, \Phi_{\mathbf{r}}^{\mathbf{v},\lambda} = -\partial_\lambda\, \Phi_{\mathbf{r}+\mathbf{m}}^{\mathbf{v},\lambda}, \quad \mathbf{m} = (m_1, m_2, m_3),$$

where $\phi_{\mathbf{m}}(\mathbf{X})$ is the marginal monomial. The first differential equation is equivalent to the identity

$$l\, m_k - \sum_{i,l_i<0} a_{ik}\, l_i = \sum_{j,l_j>0} a_{jk}\, l_j\, .$$

which is true by definition (see (4.12)).

For the second equation, using the above formulas we get that the i-th entry on the LHS is

$$-\partial_\lambda \int X_i\, \phi_{\mathbf{r}}(X)\, \frac{d^3\mathbf{X}}{dX_i} = -(r_i+1)\int \phi_{\mathbf{r}}(X)\, \frac{d^3\mathbf{X}}{dX_i},$$

where we used formulas (4.18) again. □

Let us define the row-vector $\zeta = (\zeta_1, \zeta_2, \zeta_3) = \mathbf{r}\, E_W^{-1}$. Note also that the weights (q_1^T, q_2^T, q_3^T) of the dual polynomial are precisely $(1,1,1)\, E_W^{-1}$.

Lemma IV.3. *The period integral $\Phi_{\mathbf{r}}(\sigma)$ is in the kernel of the following differential operator:*

$$\sigma^{-l}\prod_{k=0}^{l-1}(\delta-k)\prod_{i,l_i<0}\prod_{k=0}^{-l_i-1}\left(\delta+\frac{l\left(q_i^T+\zeta_i+k\right)}{l_i}\right)-C\prod_{i,l_i>0}\prod_{k=0}^{l_i-1}\left(\delta+\frac{l\left(q_i^T+\zeta_i+k\right)}{l_i}\right), \tag{4.19}$$

where $C = \prod_{i=1}^{3}(-l_i/l)^{l_i}$.

Proof. Using the second equation in Lemma IV.2 we can express the derivatives $\partial_{v_i} = v_i^{-1}\delta_i$ in terms of δ. Substituting in the first equation we get a higher order differential equation in σ only. It remains only to notice that the resulting equation is independent of v and λ. □

Picard-Fuchs equation

Let $q_0^T = 0, l_0 = -l$, and set

$$\beta_{i,k} = \frac{1}{l_i}(q_i^T+\zeta_i+k), \quad 0 \le k \le |l_i|-1. \tag{4.20}$$

The differential operator in Lemma IV.3 can be factored as the product of a Bessel differential operator

$$\prod_{i,k}(\delta + l\,\beta_{i,k}) \tag{4.21}$$

and an operator of the form

$$\prod_{i',k'}(\delta + l\beta_{i',k'}) - C\sigma^l \prod_{i'',k''}(\delta + l\beta_{i'',k''}). \tag{4.22}$$

This is done simply by factoring out the common left divisors in the two summands, i.e.,there is no pairs (i', k') and (i'', k'') in the operator (4.22), such that, $\beta_{i',k'} + 1 = \beta_{i'',k''}$.

Lemma IV.4. *The numbers* (4.20) *satisfy the following identity:*

$$\sum_{i:l_i>0}\sum_{k=0}^{l_i-1}\beta_{i,k} - \sum_{0\le j\le 3:l_j<0}\sum_{k'=0}^{-l_j-1}\left(1+\beta_{j,k'}\right) = \deg\phi_{\mathbf{r}}.$$

Proof. By definition

$$\begin{aligned} LHS &= \sum_{i:l_i>0}\left(\frac{l_i-1}{2} + q_i^T + \zeta_i\right) - \sum_{j;l_j<0}\left(-l_j - \frac{-l_j-1}{2} - q_j^T - \zeta_j\right) - \frac{l-1}{2} = \\ &= \sum_{i=0}^{3}\left(q_i^T + \zeta_i + \frac{l_i-1}{2}\right) - \frac{l-1}{2} = \sum_{i=1}^{3}\zeta_i = \deg\phi_{\mathbf{r}}. \quad \square \end{aligned}$$

The action of the operator (4.22) on a period integral is again a period integral. The latter is holomorphic at $\sigma = 0$; therefore, if it is in the kernel of the Bessel operator, it must be a polynomial in σ. But a non-zero period integral cannot be a polynomial. In other words the period $\Phi_{\mathbf{r}}(\sigma)$ is a solution to the Picard-Fuchs equation corresponding to the differential operator (4.22).

Lemma IV.5. *Let $x = C\sigma^l$; then, depending on the order of the differential operator* (4.22) *the corresponding differential equation has the form*

$$(1-x)\frac{\partial}{\partial x}\Phi_{\mathbf{r}} = \deg\phi_{\mathbf{r}}\,\Phi_{\mathbf{r}},$$

if the order is 1 and

$$x(1-x)\frac{\partial^2\Phi_{\mathbf{r}}}{\partial x^2} + (\gamma_{\mathbf{r}} - (1+\alpha_{\mathbf{r}}+\beta_{\mathbf{r}})x)\,\frac{\partial\Phi_{\mathbf{r}}}{\partial x} - \alpha_{\mathbf{r}}\beta_{\mathbf{r}}\,\Phi_{\mathbf{r}} = 0,$$

if the order is 2. In the second case we have the following identity:

$$\alpha_{\mathbf{r}} + \beta_{\mathbf{r}} - \gamma_{\mathbf{r}} = \deg\phi_{\mathbf{r}}. \tag{4.23}$$

The first part of the Lemma and the identity (4.23) are corollaries of Lemma IV.4. Unfortunately, we do not have a general combinatorial rule to determine which indexes (i', k') and (i'', k'') should appear in (4.22). In other words, the second part of the Lemma is proved on a case by case basis. It is a straightforward tedious computation. In particular, when $\phi_{\mathbf{r}} = 1$, Lemma IV.5 implies Lemma IV.1.

Solutions of hypergeometric equations

For the reader's convenience we list a solutions hypergeometric equations of the form (4.15). Let us assume that α, β, and γ are positive rational numbers. There are two cases which are used in our work.

The resonance case

We assume l is a positive integer, and

$$\gamma = \alpha + \beta = 1 - \frac{1}{l},$$

1. Near $x = 0$ the hypergeometric equation admits the following basis of solutions:

$$\begin{aligned} F_1^{(0)}(x) &= {}_2F_1(\alpha, \beta; \gamma; x), \\ F_2^{(0)}(x) &= {}_2F_1(1-\alpha, 1-\beta; 2-\gamma; x)\, x^{1-\alpha-\beta}. \end{aligned} \tag{4.24}$$

2. Near $x = 1$ a basis of solutions is given by

$$\begin{aligned} F_1^{(1)}(x) &= {}_2F_1(\alpha, \beta; 1; 1-x), \\ F_2^{(1)}(x) &= {}_2F_1(\alpha, \beta; 1; 1-x)\, \ln(1-x) + \sum_{n=1}^{\infty} b_n (1-x)^n, \end{aligned} \tag{4.25}$$

where

$$b_n = \frac{(\alpha)_n (\beta)_n}{(n!)^2} \Big(\frac{1}{\alpha} + \cdots + \frac{1}{\alpha + n - 1} + \frac{1}{\beta} + \cdots + \frac{1}{\beta + n - 1} - 2\Big(\frac{1}{1} + \cdots + \frac{1}{n}\Big)\Big). \tag{4.26}$$

3. Let us also list a basis of solutions near $x = \infty$. If $\alpha - \beta \notin \mathbb{Z}$, then

$$
(4.27) \qquad
\begin{aligned}
F_1^{(\infty)} &= x^{-\alpha}\, {}_2F_1\left(\alpha, \alpha - \gamma + 1; \alpha - \beta + 1; x^{-1}\right), \\
F_1^{(\infty)} &= x^{-\beta}\, {}_2F_1\left(\beta, \beta - \gamma + 1; \beta - \alpha + 1; x^{-1}\right).
\end{aligned}
$$

If $\alpha = \beta$, then

$$
(4.28) \qquad
\begin{aligned}
F_1^{(\infty)} &= x^{-\alpha}\, {}_2F_1\left(\alpha, \alpha - \gamma + 1; 1; x^{-1}\right), \\
F_2^{(\infty)} &= G^{2,2}_{2,2}\left(x^{-1} \,\middle|\, \begin{matrix} \gamma, 1 \\ \alpha, \beta \end{matrix} \right),
\end{aligned}
$$

where $G^{2,2}_{2,2}$ is Meijer's G-function.

The non-resonance case

Now we assume that none of the exponents

$$
(4.29) \qquad \lambda_0 = 1 - \gamma, \quad \lambda_1 = \gamma - \alpha - \beta, \quad \lambda_\infty = \beta - \alpha
$$

is an integer. Then we fix the following solutions.

1. Near $x = 0$:

$$
(4.30) \qquad
\begin{aligned}
F_1^{(0)} &= {}_2F_1\left(\alpha, \beta; \gamma; x\right), \\
F_2^{(0)} &= {}_2F_1\left(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; x\right)\, x^{1-\gamma}.
\end{aligned}
$$

2. Near $x = 1$:

$$
(4.31) \qquad
\begin{aligned}
F_1^{(1)} &= {}_2F_1\left(\alpha, \beta; \alpha + \beta - \gamma + 1; 1 - x\right), \\
F_2^{(1)} &= {}_2F_1\left(\gamma - \alpha, \gamma - \beta; \gamma - \alpha - \beta + 1; 1 - x\right)\, (1 - x)^{\gamma - \alpha - \beta}.
\end{aligned}
$$

3. Near $x = \infty$:

$$
(4.32) \qquad
\begin{aligned}
F_1^{(\infty)} &= {}_2F_1\left(\alpha, \alpha - \gamma + 1; \alpha - \beta + 1; x^{-1}\right) x^{-\alpha}, \\
F_2^{(\infty)} &= {}_2F_1\left(\beta, \beta - \gamma + 1; \beta - \alpha + 1; x^{-1}\right) x^{-\beta}.
\end{aligned}
$$

4.4 Flat coordinates

Recall $\mathscr{H}^{\text{mid}}$ is the cohomology Milnor fibration over $\mathcal{M}'$. There is an alternative way to obtain flat sections for this local system, by choosing an opposite filtration for the mixed Hodge structures and extends it to singular points, see [Mo]. Let us introduce it here. We will calculate the B-model correlators using those flat sections.

Mixed Hodge structure

There is a mixed Hodge structure on $\mathscr{H}^{\text{mid}}$, see [ArGV, V]. Briefly, the decreasing Hodge filtration $F^\bullet$ is constructed from the principal parts of all the forms of cohomological Milnor fibration. The weight filtration $\mathbb{W}_\bullet$ is constructed from local monodromy. Let us denote the local monodromy matrix by M. We can decompose M into a product of its semisimple part M_s and unipotent part M_u, $M = M_s \cdot M_u$. Let matrix N be the logarithmic part of M_u.

For any $h \in V$, let

$$l_+(h) = \min\left\{l \in \mathbb{Z} \middle| h \in \operatorname{Ker}(N^l)\right\}, \quad l_-(h) = \max\left\{l \in \mathbb{Z} \middle| h \in \operatorname{Im}(N^l)\right\}.$$

We define a weight filtration $\mathbb{W}_\bullet$ with *central index* k by

$$\mathbb{W}_{k+l} = \left\{h \middle| l_+(h) - l_-(h) \leq l + 1.\right\}$$

This is an increasing filtration

$$\{0\} = \mathbb{W}_0 \subset \ldots \cdots \subset \mathbb{W}_{2n} = \mathscr{H}^{\text{mid}}$$

Let $H^{n-1}_\lambda(X_{\mathbf{s}}, \mathbb{C})$ be the root subspace of the eigenvalue λ of the monodromy operator. Then the central index k is $n-1$ for $\lambda \neq 1$ and n for $\lambda = 1$.

A similar weight filtration can be also introduced on the homological fibration.

Opposite filtration

We define an increasing filtration $\mathbb{S}_\bullet$ on cohomology fibration by

$$\mathbb{S}_k := \mathrm{Ann}(\mathbb{W}_{2n-2k}) := \left\{ \gamma \in \mathscr{H}_{\mathrm{mid}} \middle| \int_\gamma \alpha = 0, \forall \alpha \in \mathbb{W}_{2n-2k} \right\}$$

We can choose an increasing filtration $\mathbb{S}^\bullet$, defined by

$$\mathbb{S}^q := \mathrm{Ann}(\mathbb{S}_{q-1}) = \left\{ \alpha \in \mathscr{H}^{\mathrm{mid}} \middle| \int_\gamma \alpha = 0, \forall \gamma \in \mathbb{S}_{q-1} \right\}$$

If the local monodromy is maximal unipotent, then $\mathbb{S}^\bullet$ is a *splitting filtration* for $F^\bullet$ as

$$\mathscr{H}_{\mathbf{s}}^{\mathrm{mid}} \cong F_{\mathbf{s}}^p \bigoplus \mathbb{S}_{\mathbf{s}}^{n-p+1}.$$

$\mathbb{S}_\bullet$ is also called an *opposite filtration* for $F^\bullet$ of weight n. We define

$$\mathscr{H}_{\mathbb{S}}^{p,q} = F^p \cap \mathbb{S}_q.$$

Then we have

$$\begin{cases} \mathscr{H}^{\mathrm{mid}} = \bigoplus \mathscr{H}_{\mathbb{S}}^{p,q}. \\ \\ F^p = \bigoplus_{p' \geq p} \mathscr{H}_{\mathbb{S}}^{p',n-p'}. \end{cases}$$

Now we consider sections of $\mathscr{H}_{\mathbb{S}}^{p,q}$ over an open set $U \subset \mathcal{M}'$,

$$\Gamma(U, \mathscr{H}_{\mathbb{S}}^{p,q}) := \left\{ \beta \in \Gamma(U, F^p) \middle| \int_\gamma \beta = 0, \forall \gamma \in \mathbb{S}_{q-1} \right\}.$$

Then there is a space of *distinguished sections* of $\mathscr{H}_{\mathbb{S}}^{p,q}$,

$$\Gamma(U, \mathscr{H}_{\mathbb{S}}^{p,q})_{\mathrm{dist}} := \left\{ \beta \in \Gamma(U, F^p) \middle| d(\int_\gamma \beta) = 0, \forall \gamma \in \mathbb{S}_q \right\}.$$

Choices of flat basis for sections of middle cohomology bundle

Non-twisted sectors

For an ISES W, let A be an invariant cycle in $H_1(E_\sigma)$ and B be a cycle such that A, B forms a symplectic basis for $H_1(E_\sigma)$ and

$$N(B) = A, A \cap B = 1.$$

We can check $\mathbb{S}_1$ is generated by A and $\mathbb{S}_2$ is generated by A and B. We choose cohomology sections $\int \frac{\omega}{dF}, \int \phi_{\mathbf{m}} \frac{\omega}{dF}$. Then the period matrix has a Birkhoff factorization

$$\begin{pmatrix} \int \frac{\omega}{dF} \\ \\ \int \phi_{\mathbf{m}} \frac{\omega}{dF} \end{pmatrix} \begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} \pi_A & \pi_B \\ \\ -\pi_A' & -\pi_B' \end{pmatrix} = \begin{pmatrix} \pi_A & 0 \\ \\ -\pi_A' & -\mathrm{Wr}(\pi_A, \pi_B)/\pi_A \end{pmatrix} \begin{pmatrix} 1 & \pi_B/\pi_A \\ \\ 0 & 1 \end{pmatrix}$$

Here $\mathrm{Wr}(\pi_A, \pi_B) = \pi_A \pi_B' - \pi_A' \pi_B$. We obtain a basis flat sections from non-twisted sectors

$$\begin{pmatrix} \sigma_{01} \\ \\ \sigma_{02} \end{pmatrix} = L^{-1} \begin{pmatrix} \int \frac{\omega}{dF} \\ \\ \int \phi_{\mathbf{m}} \frac{\omega}{dF} \end{pmatrix} = \begin{pmatrix} \int \frac{\omega}{dF} / \pi_A \\ \\ (-\pi_A' \int \frac{\omega}{dF} - \pi_A \int \phi_{\mathbf{m}} \frac{\omega}{dF}) / \mathrm{Wr}(\pi_A, \pi_B) \end{pmatrix}$$

It is not hard to check $\sigma_{01} \in \mathscr{H}_{\mathbb{S}}^{2,1}$ and $\sigma_{02} \in \mathscr{H}_{\mathbb{S}}^{1,2}$. We also obtain the pairing

$$\langle \sigma_{01}, \sigma_{02} \rangle = \langle \frac{\int \frac{\omega}{dF}}{\pi_A}, \frac{-\pi_A' \int \frac{\omega}{dF} - \pi_A \int \phi_{\mathbf{m}} \frac{\omega}{dF})}{\mathrm{Wr}(\pi_A, \pi_B)} \rangle = 1.$$

Twisted sectors

Let $\phi_{\mathbf{r}_1}$ be a monomial of non-integer degree. Then Gauss-Manin connection implies $\Phi_{\mathbf{r}_1} = \int \phi_{\mathbf{r}_1} \frac{\omega}{dF}$ satisfies a k-th order differential equation for some integer k. We can find other monomials $\phi_{\mathbf{r}_2}, \ldots, \phi_{\mathbf{r}_k}$ and differential operators $D_i(\sigma)$ such that for $2 \leq i \leq k$,

$$\Phi_{\mathbf{r}_i} = D_i(\sigma) \Phi_{\mathbf{r}_1}.$$

We know $\Phi_{\mathbf{r}_1}$ is a linear combination of flat sections,

$$\Phi_{\mathbf{r}_1} = f_{1,1}(\sigma) A_1 + \cdots + f_{1,k}(\sigma) A_k.$$

$f_{1,1}(\sigma), \ldots, f_{1,k}(\sigma)$ is a basis of solutions of the Picard-Fuchs equation. Apply differential operators, we have

$$\Phi_{\mathbf{r}_i} = D_i f_{1,1}(\sigma) A_1 + \cdots + D_i f_{1,k}(\sigma) A_k.$$

Now we have a flat basis $A_1, \ldots, A_k$ by inverting the $k \times k$ matrix $\left(D_i f_{1,j}(\sigma)\right)_{k \times k}$.

Remark IV.6.

- There is no canonical choices for the basis $f_{1,i}$. Especially, near different limits, we also have different choices of the basis.
- Although sections A_i are flat basis, in order to match the flat basis in A-model side, we may have to rescale the basis by some constants.
- Mirror symmetry is a computational result, rather than a conceptional one.
- All the data we need for numerical results are the indices $\alpha, \beta, \ldots, \gamma$ from the corresponding hypergeometric equation. Some of them are of order three. But none of them will have order greater than three. In this paper, all the examples we compute have order at most two. Those cases of order three will appear somewhere else.

B-model 3,4-point genus-0 correlators

We let $W = M_1 + M_2 + M_3$ be an ISES with a miniversal deformation given by a monomial $\phi_{\mathbf{m}}(\mathbf{X})$, $\mathbf{m} = (m_1, m_2, m_3)$. We choose a primitive form $\omega = d^3\mathbf{X}/\pi(\sigma)$ in a neighborhood of $\sigma = 0$, such that $\pi(\sigma)$ is the solution to the Picard-Fuchs equation (4.14) satisfying the initial conditions $\pi(0) = c, \pi'(0) = 0$, where the constant c is such that the residue pairing (see (4.4)) satisfies

$$\langle 1, \phi_{\mathbf{m}} \rangle|_{\mathbf{s}=0} = 1 \,.$$

Let $\{t_{\mathbf{r}}\}$ be the flat coordinate system, such that $t_{\mathbf{r}}(0) = 0$ and the flat vector fields $\partial_{\mathbf{r}} := \partial/\partial t_{\mathbf{r}}$ agree with $\partial/\partial s_{\mathbf{r}}$ at $\mathbf{s} = 0$.

The primitive form induces an isomorphism between the tangent and the vanishing cohomology bundle via the following period mapping:

$$\partial/\partial t_{\mathbf{r}} \mapsto -\nabla^{-1}_{\frac{\partial}{\partial\lambda}} \nabla_{\frac{\partial}{\partial t_{\mathbf{r}}}} \int \frac{\omega}{dF} = \int \delta_{\mathbf{r}}(\mathbf{s}, \mathbf{X}) \, \frac{\omega}{dF}, \tag{4.33}$$

where $\delta_{\mathbf{r}}$ is some homogeneous polynomial (in $\mathbf{X}$) of degree $\deg(\phi_{\mathbf{r}})$. The images are flat distinguished sections of opposite filtrations. Note that the Kodaira–Spencer isomorphism takes the form

$$\partial/\partial t_{\mathbf{r}} \mapsto \delta_{\mathbf{r}}(\mathbf{s}, \mathbf{X}) \bmod (F_{X_1}, F_{X_2}, F_{X_3}). \tag{4.34}$$

By definition, the restriction of the 3-point correlators to the marginal deformations subspace is

$$\langle \delta_{\mathbf{r}_1}, \delta_{\mathbf{r}_2}, \delta_{\mathbf{r}_3} \rangle_{0,3} = \operatorname{Res} \frac{\delta_{r_1}(\sigma, \mathbf{X})\delta_{r_2}(\sigma, \mathbf{X})\delta_{r_3}(\sigma, \mathbf{X})}{(\partial_{X_1} W_\sigma)\,(\partial_{X_2} W_\sigma)(\partial_{X_3} W_\sigma)}\, \frac{d^3\mathbf{X}}{\pi(\sigma)^2}. \tag{4.35}$$

Note that the 3-point correlator depends only on the product $\Xi := \delta_{\mathbf{r}_1}\delta_{\mathbf{r}_2}\delta_{\mathbf{r}_3}$. Therefore we can simply use the notation $\langle \Xi \rangle_{0,3}$ instead. Finally, definition (4.35) makes sense even if we replace $\delta_{\mathbf{r}}$, $\mathbf{r} = \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ by arbitrary polynomials, not only the ones that correspond to flat vector fields via (4.34).

Let $\mathscr{F}_0^{SG}$ be the genus-0 generating functions for the Frobenius manifold of miniversal deformations near the origin. By definition,

$$\langle \delta_{\mathbf{r}_1}, \ldots, \delta_{\mathbf{r}_n} \rangle_{0,n}^{SG} = \frac{\partial^n \mathscr{F}_0^{SG}}{\partial\, t_{\mathbf{r}_1} \ldots \partial\, t_{\mathbf{r}_n}}\Big|_{\mathbf{t}=0}.$$

Thus, using that $\partial/\partial\sigma = \delta_{\mathbf{m}}$ at $\sigma = 0$, we can compute the Saito's genus-0 4-point correlators with a top degree flat insertion by the following formula

$$\langle \delta_{\mathbf{r}_1}, \delta_{\mathbf{r}_2}, \delta_{\mathbf{r}_3}, \delta_{\mathbf{m}} \rangle_{0,4}^{SG} = \partial_\sigma \left\langle \delta_{\mathbf{r}_1}, \delta_{\mathbf{r}_2}, \delta_{\mathbf{r}_3} \right\rangle\Big|_{\sigma=0}. \tag{4.36}$$

4.5 Givental's higher genus formula at semisimple points and its extension

In Chapter VII, we will define Givental's higher genus formula for singularity theory at a semisimple point $\mathbf{t} \in \mathcal{M}$, we call it the *formal* ancestor potential $\mathcal{A}^{formal}(\mathbf{t})(\hbar, \mathbf{q}(z))$. Here is a explicit formula, for more details, see Chapter VII.

$$\widehat{\Psi}(\mathbf{t})\, \widehat{R}(\mathbf{t})\, e^{(U(\mathbf{t})/z)^\wedge} \prod_{i=1}^{\mu} \mathcal{A}^{\mathrm{pt}}(\hbar\Delta_i(\mathbf{t}), {}^i\widetilde{\mathbf{q}}(z)\, \sqrt{\Delta_i(\mathbf{t})}),$$

In this formula, $\Psi(\mathbf{t}), R(\mathbf{t}), U(\mathbf{t}), \Delta_i(\mathbf{t})$ are all data from a singularity W, and the symbol $\frown$ is quantization operator. We will be interested in the limit when $\mathbf{t}$ is a special point, which is not semisimple. So the formula is not well defined at those points. However, our convergence theorems will guarantee that the limit of those formula exists. So we can study the mirror symmetry where the B-model potentials are those limits. Details about the mirror theorems will be discussed in later chapters.

CHAPTER V

Berglund-Hübsch-Krawitz mirror construction

5.1 BHK construction

Now let us start to describe the Berglund-Hübsch-Krawitz mirror construction. We begin with a polynomial with N variables and N monomials.

$$W = \sum_{i=1}^{N} \prod_{j=1}^{N} X_i^{a_{ij}}.$$

We denote its *exponent matrix* by E_W, where the (i, j)-th entry of E_W is a_{ij}. We say this polynomial W is invertible if its exponent matrix E_W is an invertible matrix. We consider the *transpose* of W as the polynomial whose exponent matrix is the transpose matrix of E_W, and denote it by W^T. Then

$$W^T = \sum_{i=1}^{N} \prod_{j=1}^{N} X_i^{a_{ji}}.$$

This is considered to be the mirror for W [BH].

Now we introduce the mirror group construction defined by Krawitz, [Kr]. Let us write the inverse matrix E_W^{-1} with row vectors ρ_i, such that ρ_i is the i-th row of E_W^{-1}. We denote each ρ_i by

$$\rho_i = (\rho_{i,1}, \ldots, \rho_{i,N})$$

We consider a diagonal matrix E_{ρ_i}, with its j-th diagonal entry $\exp(2\pi\sqrt{-1}\rho_{i,j})$. It turns out each E_{ρ_i} is a symmetry of W^T, i.e. the matrix E_{ρ_i} acts on $(X_1, \ldots, X_N)$ by multiplication will keep W^T invariant. Let us also point out that E_{ρ_i} generate G_{W^T}, the maximal diagonal symmetry group of W^T.

On the other hand, we can also consider the columns of E_W^{-1}. We denote its i-th column by ϱ_i. We get another diagonal matrix E_{ϱ_i}, with its j-th diagonal entry defined by the exponental of $2\pi\sqrt{-1}$ multiplies the j-th element of ϱ_i. E_{ϱ_i} acts on variables $X_1, \ldots, X_N$ and keeps W invariant. Similarly, we know E_{ϱ_i} generate G_W. Krawitz [Kr] defined a mirror group G^T by

$$G^T := \left\{ \prod_{i=1}^{N} E_{\rho_i}^{k_i} \middle| (k_1, \ldots, k_N)\, E_W^{-1}\, (m_1 \ldots m_N)^T \in \mathbb{Z}, \forall \prod_{i=1}^{N} E_{\varrho_i}^{m_i} \in G \right\} \tag{5.1}$$

Such a construction (W^T, G^T) is considered to be the Berglund-Hübsch-Krawitz mirror of a pair (W, G). Here is a very useful observation from [Kr]. The mirror group for G_W is the trivial group with the identity matrix. We simply denote by

$$G_W^T = \{1\}.$$

In this paper, we only consider the case that G_{W^T} is the admissible group in the FJRW theory. So its BHK mirror would just be W with the trivial group.

5.2 Classification of ring structures

For any special limit in the Saito-Givental theory for invertible simple elliptic singularities, the first step of the classification is the ring structure. Let us first focus on special limits at $\sigma = 0$. According to Saito [Sa2] simple elliptic singularities are classified by their Milnor number and the elliptic curve at infinity. It follows that the Jacobi algebras of the ISES with 3 variables can be classified into 6 isomorphism classes parametrized by the pairs consisting of the Milnor number $\mu = \dim \mathcal{Q}_W$ and the j-invariant of $E_{\sigma=0}$:

Table 5.1: Classification of Jacobi algebra

μ	$j(\sigma = 0)$	W
8	0	$X_1^3 + X_2^3 + X_3^3, \quad X_1^2X_2 + X_2^3 + X_3^3, \quad X_1^2X_2 + X_1X_2^2 + X_3^3, \quad X_1^2X_2 + X_2^2X_3 + X_1X_3^2$
8	1728	$X_1^2X_2 + X_2^2X_3 + X_3^3$
9	0	$X_1^3X_2 + X_2^4 + X_3^2, \quad X_1^3X_2 + X_2^2X_3 + X_3^2$
9	1728	$X_1^4 + X_2^4 + X_3^2$
10	0	$X_1^6 + X_2^3 + X_3^2, \quad X_1^3X_2 + X_2^2 + X_3^3$
10	1728	$X_1^4X_2 + X_2^3 + X_3^2$

For any two polynomials in the same list, it is easy to find a linear map between the generators X_1, X_2, X_3 of the corresponding Jacobi algebras, such that it induces a ring isomorphism. Let us point out that the choice of such linear maps is not unique in general. Moreover, we can always adjust some constants such that the ring isomorphism will be extended to an isomorphism of Frobeniu manifold, as well as an isomorphism of the corresponding ancestor total potential. See the discussions in Section 3.4.

5.3 Mirror to Genper point

In order to compute 4-point correlators of the form (4.36) it is enough to determine $\delta_{\mathbf{r}}(\sigma, \mathbf{X})$ up to linear terms in σ. To begin with, we notice that $\phi_{\mathbf{r}+\mathbf{m}}$ lies in the Jacobian ideal of W_σ. More precisely, the following Lemma holds.

Lemma V.1. *There are polynomials $g_{\mathbf{r},i} \in \mathbb{C}[\sigma, X_1, X_2, X_3]$ such that*

$$(1 - C\sigma^l)\, \phi_{\mathbf{r}+\mathbf{m}} = \sum_{i=1}^{3} g_{\mathbf{r},i}\, \partial_i W_\sigma.$$

This Lemma can be proved in all cases by using Saito's higher residue pairing. However, in what follows, we need an explicit formula for

$$g_{\mathbf{r}} := (g_{\mathbf{r},1}, g_{\mathbf{r},2}, g_{\mathbf{r},3})\,.$$

Therefore we verified the lemma on a case-by-case basis. Some of our computations will be given below. The remaining cases are completely analogous.

There are several corollaries of Lemma V.1. First of all, note that under the period map (4.33) the Gauss–Manin connection takes the form (4.8) (with $z \equiv -\partial_\lambda^{-1}$. It follows that if $\deg(\phi_{\mathbf{r}})$ is not integral, then the restriction of the section (4.33) of the vanishing cohomology bundle to the marginal deformation subspace must be flat, i.e., the sections

$$[\delta_{\mathbf{r}}\,\omega](\sigma) := \int \delta_{\mathbf{r}}(\sigma, \mathbf{X})\,\frac{\omega}{dW_\sigma}, \quad \deg(\phi_{\mathbf{r}}) \notin \mathbb{Z} \tag{5.2}$$

are independent of σ. Furthermore, using formulas (4.18) for the Gauss-Manin connection we get

$$(1 - C\sigma^l)\frac{\partial}{\partial\sigma}\Phi_{\mathbf{r}} = -\int \sum_{i=1}^{3} \partial_i\, g_{\mathbf{r},i}\,\frac{d^3\mathbf{X}}{dW_\sigma}$$

Both sides must have the same degree, i.e.,

$$(1 - C\sigma^l)\frac{\partial}{\partial\sigma}\Phi_{\mathbf{r}} = \sum_{\mathbf{r}'} c_{\mathbf{r},\mathbf{r}'}(\sigma)\Phi_{\mathbf{r}'}, \tag{5.3}$$

where the sum is over all $\mathbf{r}'$, such that $\deg\phi_{\mathbf{r}} = \deg\phi_{\mathbf{r}'}$ and $c_{\mathbf{r},\mathbf{r}'}(\sigma) \in \mathbb{C}[\sigma]$ are some polynomials.

Lemma V.2. *Suppose* $\deg(\phi_{\mathbf{r}}) \notin \mathbb{Z}$*; then we have*

$$\delta_{\mathbf{r}} = \phi_{\mathbf{r}} - \sigma \sum_{\mathbf{r}',\mathbf{r}'\neq\mathbf{r}} c_{\mathbf{r},\mathbf{r}'}(0)\,\phi_{\mathbf{r}'} + O(\sigma^2), \tag{5.4}$$

where $O(\sigma^2)$ *denotes terms that have order of vanishing at* $\sigma = 0$ *at least 2.*

Proof. Follows easily from (5.3). We omit the details. □

Let $M \in \mathbb{C}[\mathbf{X}]$ be a weight-1 monomial with leading coefficient 1. Our next goal is to evaluate the following auxiliar expression:

$$\langle M, \phi_{\mathbf{m}}\rangle_{0,4} := \partial_\sigma \langle M\rangle_{0,3}\Big|_{\sigma=0}.$$

Lemma V.3. *The number* $\langle M, \phi_{\mathbf{m}}\rangle_{0,4}$ *is non-zero iff* $M = M_i$ *for some* $i = 1, 2, 3$. *In the latter cases the numbers are given as follows*

$$\Big(\langle M_1, \phi_{\mathbf{m}}\rangle_{0,4}, \langle M_2, \phi_{\mathbf{m}}\rangle_{0,4}, \langle M_3, \phi_{\mathbf{m}}\rangle_{0,4}\Big) = -(m_1, m_2, m_3)E_W^{-1}. \tag{5.5}$$

Proof. For the second part, we apply the operators $X_i\,\partial_{X_i}$, $i = 1, 2, 3$, to the identity

$$M_1 + M_2 + M_3 = W_\sigma - \sigma\phi_{\mathbf{m}}(\mathbf{X})$$

and take the residue. We get

$$\langle M_1\rangle_{0,3}\, a_{1i} + \langle M_2\rangle_{0,3}\, a_{2i} + \langle M_3\rangle_{0,3}\, a_{3i} = -\sigma\, m_i\, \langle \phi_{\mathbf{m}}\rangle_{0,3}.$$

It remains only to differentiate with respect to σ and set $\sigma = 0$.

For the first part, because M is a weight-1 monomial with coefficient 1, we can use the relations in the Jacobi algebra of W_σ to rewrite M as a product of $\phi_{\mathbf{m}}$ and a function of σ. Let us write $M = h(\sigma)\,\phi_{\mathbf{m}}$. For example, in the Fermat $E_6^{(1,1)}$ case,

$$X_1^3 = -3\sigma\phi_{111}; \quad (1 + \frac{\sigma^3}{27})\, X_1^2 X_2 = 0.$$

If $M \neq M_i$, $i = 1, 2, 3$, then $h(\sigma)$ either does not vanish at $\sigma = 0$ or vanishes at $\sigma = 0$ with order at least 2. In both cases, $\langle M, \phi_{\mathbf{m}}\rangle_{0,4}$ vanish. □

Now we are ready to compute the 4-point correlators that are needed for the reconstruction of the CohFT. Let $\delta_{\mathbf{r}}(\mathbf{s}, \mathbf{X})$, $\mathbf{r} = \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ be polynomials corresponding to the flat vector fields $\partial/\partial t_{\mathbf{r}}$ via the Kodaira–Spencer isomorphism (4.34). Put

$$\Xi(\mathbf{s}, \mathbf{X}) = \delta_{\mathbf{r}_1}(\mathbf{s}, \mathbf{X})\delta_{\mathbf{r}_2}(\mathbf{s}, \mathbf{X})\delta_{\mathbf{r}_3}(\mathbf{s}, \mathbf{X}).$$

Note that $\Xi(0, \mathbf{X})$ is a homogeneous monomial (see (5.4)) with leading coefficient 1.

Lemma V.4. *The* 4*-point genus-*0 *correlators with a top degree insertion* $\delta_{\mathbf{m}}$ *are*

$$\langle \delta_{\mathbf{r}_1}, \delta_{\mathbf{r}_2}, \delta_{\mathbf{r}_3}, \delta_{\mathbf{m}}\rangle_{0,4}^{SG} = \begin{cases} -q_i^T & , \text{ if } \Xi(0, \mathbf{X}) = M_i, \\ 0 & , \text{ otherwise.} \end{cases}$$

Proof. The same argument in first part of Lemma V.3 also works for $\Xi(\sigma, \mathbf{X})$. Thus if $\Xi \neq M_i$, i=1,2,3, we have

$$\langle \Xi, \delta_{\mathbf{m}}\rangle_{0,4}^{SG} = 0.$$

In order to finish the proof we need only to compute the correlators when $\Xi(0,\mathbf{X}) = M_i$ for some $i = 1,2,3$. Note that the diagonal entries of the matrix E_W are always at least 2 (see Table 1.1). Therefore, it is enough to compute the following correlators:

$$\begin{cases} \langle \delta_{100}, \delta_{100}, \delta_{\mathbf{r}}, \delta_{\mathbf{m}} \rangle_{0,4}^{SG}, & \mathbf{r} = (a_{11}-2, a_{12}, a_{13}), \\ \langle \delta_{010}, \delta_{010}, \delta_{\mathbf{r}}, \delta_{\mathbf{m}} \rangle_{0,4}^{SG}, & \mathbf{r} = (a_{21}, a_{22}-2, a_{23}), \\ \langle \delta_{001}, \delta_{001}, \delta_{\mathbf{r}}, \delta_{\mathbf{m}} \rangle_{0,4}^{SG}, & \mathbf{r} = (a_{31}, a_{32}, a_{33}-2). \end{cases}$$

We do not have a uniform computation since we need to use Lemma V.2, for which the coefficients $c_{\mathbf{r},\mathbf{r}'}(0)$ can be computed only on a case-by-case basis. Let us sketch the main steps of the computation in several examples, leaving the details and the remaining cases to the reader. We will make use of the notation

$$\delta(\sigma,\mathbf{X}) \approx \phi(\sigma,\mathbf{X}), \quad \delta,\phi \in \mathbb{C}[\mathbf{X}],$$

which means first order approximation at $\sigma = 0$, i.e., $\delta(\sigma,\mathbf{X}) - \phi(\sigma,\mathbf{X}) = O(\sigma^2)$.

Case 1: $W = X_1^3+X_2^3+X_3^3 \in E_6^{(1,1)}$ and $\phi_{\mathbf{m}} = X_1X_2X_3$. Since W is symmetric in X_1, X_2, X_3 it is enough to compute only one of the correlators, say $\Xi = M_1$. After a straightforward computation (the notation is the same as in Lemma V.1) we get

$$g_{100} = \Big(\frac{1}{3}\,\phi_{011}, -\frac{\sigma}{9}\,\phi_{002}, \frac{\sigma^2}{27}\,\phi_{101}\Big).$$

It follows that $\delta_{100} \approx \phi_{100}$ and then using formula (5.5) we get

$$\langle \delta_{100}, \delta_{100}, \delta_{100}, \delta_{\mathbf{m}} \rangle_{0,4}^{SG} = -\frac{1}{3}.$$

Case 2: $W = X_1^4 + X_2^4 + X_3^2 \in E_7^{(1,1)}$ and $\phi_{\mathbf{m}} = X_1^2X_2^2$. In this case $M_3 = 0$ in the Jacobi algebra of W and W is symmetric in X_1 and X_2. It is enough to compute only one of the correlators, say the one with $\Xi(0,\mathbf{X}) = M_1$. We have

$$g_{100} = \Big(\frac{1}{4}\,\phi_{020}, -\frac{\sigma}{8}\,\phi_{110}, 0\Big).$$

It follows that

$$\delta_{100} \approx \phi_{100}, \quad \delta_{200} \approx \phi_{200} + \frac{\sigma}{4}\phi_{020}.$$

Using formula (5.5) we find

$$\langle \delta_{100}, \delta_{100}, \delta_{400}, \delta_{\mathbf{m}} \rangle_{0,4}^{SG} = -\frac{1}{4}.$$

Case 3: $W = X_1^3 X_2 + X_2^2 + X_3^3 \in E_8^{(1,1)}$ and $\phi_{\mathbf{m}} = X_1 X_2 X_3$. In this case, since $M_2 = 0$ in the Jacobi algebra, we need to compute two correlators. We have

$$\begin{pmatrix} g_{100} \\ \\ g_{010} \\ \\ g_{001} \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\phi_{001} - \frac{\sigma^2}{54}\phi_{200} & \frac{\sigma^2}{18}\phi_{110} & -\frac{\sigma}{9}\phi_{010} \\ \\ -\frac{1}{6}\phi_{201} + \frac{\sigma^2}{27}\phi_{110} & \frac{1}{2}\phi_{111} & -\frac{\sigma^2}{9}\phi_{210} \\ \\ -\frac{\sigma}{9}\phi_{010} - \frac{\sigma^2}{54}\phi_{101} & \frac{\sigma^2}{9}\phi_{011} & \frac{1}{3}\phi_{110} \end{pmatrix}.$$

It follows that we have the following linear approximations:

$$\delta_{100} \approx \Phi_{100}, \quad \delta_{001} \approx \Phi_{001}, \quad \delta_{110} \approx \Phi_{110}.$$

The correlators then become

$$\langle \delta_{100}, \delta_{100}, \delta_{110}, \delta_{\mathbf{m}} \rangle_{0,4}^{SG} = -\frac{1}{3}, \quad \langle \delta_{001}, \delta_{001}, \delta_{001}, \delta_{\mathbf{m}} \rangle_{0,4}^{SG} = -\frac{1}{3}$$

Case 4: The Fermat type $E_8^{(1,1)}$, i.e. $W = X_1^6 + X_2^3 + X_3^2$ and $\phi_{\mathbf{m}} = X_1^4 X_2$. In this case $M_3 = 0$, so again we have to compute two correlators. We have

$$g_{100} = \Big(\frac{1}{6}\phi_{020} + \frac{\sigma^2}{27}\phi_{200}, -\frac{2\sigma}{9}\phi_{300}, 0\Big), \quad g_{010} = \Big(-\frac{\sigma}{18}\phi_{300} + \frac{\sigma^2}{27}\phi_{110}, \frac{1}{3}\phi_{400}, 0\Big).$$

It follows that the first order approximations that we need are

$$\delta_{100} \approx \Phi_{100}, \quad \delta_{010} \approx \Phi_{010}, \quad \delta_{400} \approx \Phi_{400} + \frac{\sigma}{2}\Phi_{210}.$$

Formulas (5.5) and (4.36) imply

$$\langle \delta_{100}, \delta_{100}, \delta_{400}, \delta_{\mathbf{m}} \rangle_{0,4}^{SG} = -\frac{1}{6}; \quad \langle \delta_{010}, \delta_{010}, \delta_{010}, \delta_{\mathbf{m}} \rangle_{0,4}^{SG} = -\frac{1}{3}. \quad \square$$

Comparing the 4-point correlators in Lemmas III.10 and V.4 we see that they have opposite signs. It is not hard to see that if we rescale the primitive form by (-1); then the 3-point correlators in the Saito–Givental theory do not change, while the 4-point ones change their sign. Therefore, we rescale the primitive form by -1 and define the map

$$\mathscr{H}_{W^T} \to T\mathcal{M}_W, \quad \rho_{\mathbf{r}} \mapsto (-1)^{1-\deg\phi_{\mathbf{r}}} \delta_{\mathbf{r}}, \quad \mathbf{r} = (r_1, r_2, r_3),$$

where $\rho_{\mathbf{r}} = \rho_1^{r_1} \rho_2^{r_2} \rho_3^{r_3}$. Lemmas III.10 and V.4 imply that the above map is a mirror symmetry map, i.e., it identifies the correlators of the FJRW theory of (W^T, G_{W^T}) and the correlators of the Saito-Givental theory of W. Theorem I.5 is proved.

CHAPTER VI

Global mirror symmetry

In this chapter, we give some examples of the classification of special points.

Construction of the mirror map

The primitive form is chosen to be $\omega = d^3\mathbf{X}/\pi_A(\sigma)$, where the cycle $A \in H_1(E_\sigma)$ is invariant with respect to the local monodromy around $\sigma = p_k$. Recall that $\pi_A(\sigma)$ is a solution to the Picard–Fuchs equation (IV.1). The latter has near $\sigma = p_k$ a basis of solutions $\{F_1^{(1)}(x), F_2^{(2)}(x)\}$ given by formula (4.25). The invariance of A implies that

$$\pi_A(\sigma) = \lambda_W \, F_1^{(1)}(x) \tag{6.1}$$

for some scalar $\lambda_W \in \mathbb{C}^*$. We choose a second cycle $B \in H_1(E_\sigma)$, such that

$$\tau := \frac{\pi_B(\sigma)}{\pi_A(\sigma)}$$

is the modulus of the elliptic curve E_σ; then we have

$$\pi_B(\sigma) = \frac{K\lambda_W}{2\pi\sqrt{-1}} \left(F_2^{(1)}(x) + K' \cdot F_1^{(1)}(x)\right), \tag{6.2}$$

where K, K' are constants whose values are given as follows. The j-invariant of E_σ has the form

$$j(\sigma) = \frac{P(\sigma)}{(1 - C\sigma^l)^N}, \quad P(\sigma) \in \mathbb{C}[\sigma] \tag{6.3}$$

for some polynomial $P(\sigma)$ and some integer N. Then

$$K = N, \quad K' = -\frac{1}{N} \ln P(p_k).$$

Note that

$$\langle 1, \phi_{-1} \rangle_A = \frac{1}{K''(1 - C\,\sigma^l)\,\pi_A^2}, \tag{6.4}$$

where K'' is some constant and the index A on the LHS indicates that we are computing the residue pairing (4.4) with respect to the primitive form ω. Since the residue pairing must be identified with the Poincaré pairing, the mirror map should satisfy

$$\Delta_{01} \mapsto 1, \quad \Delta_{02} \mapsto K''(1 - C\,\sigma^l)\phi_{-1}(\mathbf{X})\pi_A^2. \tag{6.5}$$

The next step is to identify the divisor coordinate t_{02} in the orbifold GW theory and the modulus τ. We define

$$t_{02} = t_{-1} := \frac{2\pi\sqrt{-1}}{L}\tau, \tag{6.6}$$

where $L = 3, 4, 6$ respectively for the elliptic orbifolds $\mathbb{P}^1_{3,3,3}, \mathbb{P}^1_{4,4,2}, \mathbb{P}^1_{6,3,2}$.

Since

$$1 = \langle \Delta_{01}, \Delta_{02} \rangle = \langle 1, \frac{\partial}{\partial t_{-1}} \rangle = \frac{\partial \sigma}{\partial \tau}\frac{\partial \tau}{\partial t_{-1}} \langle 1, \frac{\partial}{\partial \sigma} \rangle_A = \frac{\partial \sigma}{\partial \tau}\frac{\partial \tau}{\partial t_{-1}} \langle 1, \phi_{-1} \rangle_A,$$

we get

$$\frac{1}{K''(1 - C\,\sigma^l)\,\pi_A^2} = \frac{2\pi\sqrt{-1}}{L}\frac{\partial \tau}{\partial \sigma} = \frac{2\pi\sqrt{-1}}{L}\frac{\pi_B'\,\pi_A - \pi_B\,\pi_A}{\pi_A^2}.$$

Note that $\pi_B'\,\pi_A - \pi_B\,\pi_A'$ is the Wronskian of the Picard–Fuchs equation (IV.1), so it must be proportional to $(1 - C\,\sigma^l)^{-1}$. The proportionality coefficient can be found by comparing the Laurent series expansions at $\sigma = p_k$. Namely, it is $-\frac{N}{2\pi i}\,\lambda_W^2 l\, C^{1/l}\eta^k$. This determines the value of λ_W^2,

$$\lambda_w^2 = -\frac{L}{NK''lC^{1/l}\eta^k}. \tag{6.7}$$

The most delicate part of constructing the mirror map is finished. In order to complete the construction, we need to identify the twisted cohomology classes Δ_{ij} with monomials $\delta_{\mathbf{r}}(\sigma, \mathbf{X})$. The key observation is that the sections

$$\int \delta_{\mathbf{r}}(\sigma, \mathbf{X}) \frac{\omega}{dW_\sigma} \tag{6.8}$$

of the vanishing cohomology bundle of W_σ are flat with respect to the Gauss–Manin connection. This way our choice of $\delta_{\mathbf{r}}$ depends on an invertible matrix of size $(\mu-2)\times(\mu-2)$. The correlation functions in the Saito–Givental CohFT are invariant with respect to the translation $t_{-1} \mapsto t_{-1} + 2\pi\sqrt{-1}$, i.e., we can expand the correlation functions into Fourier series in $q := e^{t_{-1}}$. The coefficient in front of q^d, $d \in \mathbb{Z}$, is called the *degree-d* part of the correlator function. By taking the degree-0 part of the 3-point functions, we obtain a Frobenius algebra structure on the Jacobi algebra $\mathcal{Q}_W$ that under the mirror map should be identified with the Frobenius algebra corresponding to the Chen–Ruan orbifold (classical) cup product. Using also that the mirror map preserves homogeneity we obtain a system of equations for the matrix. It remains only to see that these equations have a solution. Let us list the explicit formulas for the mirror map for several examples. We omit the details of the computations, which by the way are best done with the help of some computer software (e.g. Mathematica or Maple).

6.1 Mirror to LCSL point at roots of unity

Large complex structure limit point for $W_\sigma = X_1^3 + X_2^3 + X_3^3 + \sigma X_1X_2X_3$

For this example $x = -\sigma^3/27$, i.e., $C = -1/27$ and $l = 3$. The j-invariant is

$$j(\sigma) = -\frac{\sigma^3(-216+\sigma^3)^3}{(27+\sigma^3)^3} = \frac{-27x(8+x)^3}{(1-x)^3}.$$

We have $p_l = -3$ and

$$K = N = 3, \quad \Delta_{02} = 27(1-x)X_1X_2X_3\pi_A^2,$$

which implies that $\lambda_W = \pm 1$. We pick $\lambda_W = 1$.

The Fourier series of $1 - x$ in $q = e^{2\pi i\tau/3}$ is

$$1 - x = -27q - 324q^2 - 2430q^3 - 13716q^4 + \cdots \tag{6.9}$$

Let

$$\mathbf{r} = (1,0,0), (0,1,0), (0,0,1); \quad \mathbf{r}' = (1,1,1) - \mathbf{r}.$$

A natural basis for the flat sections with non-integral degrees are

$$\delta_{\mathbf{r}} = (1-x)^{1/3}\phi_{\mathbf{r}}(\mathbf{X})\,\pi_A, \quad \delta_{\mathbf{r}'} = (1-x)^{2/3}\phi_{\mathbf{r}'}(\mathbf{X})\,\pi_A.$$

Applying (4.35), we know the non-vanishing correlators $\langle \cdots \rangle_{0,3,0}$ are

$$\langle 1, \delta_{\mathbf{r}}, \delta_{\mathbf{r}'} \rangle_{0,3,0} = \langle \delta_{\mathbf{r}}, \delta_{\mathbf{r}}, \delta_{\mathbf{r}} \rangle_{0,3,0} = \langle \delta_{1,0,0}, \delta_{0,1,0}, \delta_{0,0,1} \rangle_{0,3,0} = \frac{1}{27}.$$

The mirror map is given by (6.5), (6.6), and it identifies the ring generators as follows:

$$\begin{pmatrix} \Delta_{11} \\ \Delta_{21} \\ \Delta_{31} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \mathbf{e}[\frac{2}{3}] & \mathbf{e}[\frac{1}{3}] \\ 1 & \mathbf{e}[\frac{1}{3}] & \mathbf{e}[\frac{2}{3}] \end{pmatrix} \begin{pmatrix} \delta_{1,0,0} \\ \delta_{0,1,0} \\ \delta_{0,0,1} \end{pmatrix}.$$

It is easy to check that this identification agrees with the Chen-Ruan orbifold cohomology ring of $\mathbb{P}^1_{3,3,3}$ (see Chapter II). For example, we have

$$\langle \Delta_{11}, \Delta_{11}, \Delta_{11} \rangle_{0,3,0} = \sum_{\mathbf{r}; \deg \phi_{\mathbf{r}} = 1/3} \langle \delta_{\mathbf{r}}, \delta_{\mathbf{r}}, \delta_{\mathbf{r}} \rangle_{0,3,0} + 6 \langle \delta_{1,0,0}, \delta_{0,1,0}, \delta_{0,0,1} \rangle_{0,3,0} = \frac{1}{3}.$$

Now we need only to check that

$$\langle \Delta_{11}, \Delta_{21}, \Delta_{31} \rangle_{0,3,1} = 1.$$

After a straightforward computation we get

$$\langle \Delta_{11}, \Delta_{21}, \Delta_{31} \rangle_{0,3} = \frac{(-\sigma - 3)\pi_A}{27} = \frac{(1-(1-x))^{1/3} - 1}{9}\pi_A = q + q^4 + 2q^7 + \cdots$$

Large complex structure limit point for $W_\sigma = X_1^4 + X_2^4 + X_3^2 + \sigma X_1^2 X_2^2$

We substitute $x = \sigma^2/4$, i.e., $l = 2$ and $C = 1/4$. The j-invariant is

$$j(\sigma) = \frac{16(12+\sigma^2)^3}{(4-\sigma^2)^2} = \frac{64(3+x)^3}{(1-x)^2}.$$

We have $p_2 = 2$ and

$$K = N = 2, \quad \Delta_{02} = 16(1-x)\, X_1^2 X_2^2\, \pi_A^2.$$

It follows that $\lambda_W = \frac{\sqrt{2}}{4}$. The Fourier series for $1-x$ in terms of $q = e^{2\pi i\tau/4}$ is

$$1 - x = 64q^2 - 1536q^4 + 19200q^6 + \cdots .$$

Let us construct a basis of flat sections. First, note that the periods

$$\Phi_{\mathbf{r}}, \quad \mathbf{r} = (10, 01, 11, 21, 12)$$

still satisfy first order differential equations and that the corresponding flat sections will be

$$\delta_{\mathbf{r}}(\sigma, \mathbf{X}) = (x-1)^{\deg\phi_{\mathbf{r}}} \phi_{\mathbf{r}}(\mathbf{X})\, \pi_A.$$

However, both Φ_{20} and Φ_{02} satisfy a second order hypergeometric equation with parameters respectively $\alpha_{20} = \alpha_{02} = 3/4, \beta_{20} = \beta_{02} = 1/4, \gamma_{20} = \gamma_{02} = 1/2$. Namely, the periods satisfy the following system:

$$\begin{cases} (4-\sigma^2)\,\partial_\sigma \Phi_{20}(\sigma) = \dfrac{\sigma}{2}\,\Phi_{20}(\sigma) - \Phi_{02}(\sigma); \\ (4-\sigma^2)\,\partial_\sigma \Phi_{02}(\sigma) = \dfrac{\sigma}{2}\,\Phi_{02}(\sigma) - \Phi_{20}(\sigma). \end{cases} \tag{6.10}$$

It follows that $\Phi_{02} = L\,\Phi_{20}$ (and $\Phi_{20} = L\,\Phi_{02}$) where L is the differential operator

$$L = -(4-\sigma^2)\partial_\sigma + \frac{\sigma}{2},$$

which lead to second order differential equations. Let us denote by $\left\{F^{(1)}_{1,\mathbf{r}}, F^{(1)}_{2,\mathbf{r}}\right\}$ the basis of solutions (4.25) of the hypergeometric equations near $x = 1$ for the weights $(\alpha_{\mathbf{r}}, \beta_{\mathbf{r}}, \gamma_{\mathbf{r}})$.

Thus we can obtain a pair of polynomials δ_{20} and δ_{02} that determine flat sections by solving the following system:

$$\begin{pmatrix} \phi_{20}\,\pi_A \\ \\ \phi_{02}\,\pi_A \end{pmatrix} = \begin{pmatrix} c_{20,1}\,F^{(1)}_{1,20} & c_{20,2}\,F^{(1)}_{2,20} \\ \\ L(c_{20,1}\,F^{(1)}_{1,20}) & L(c_{20,2}\,F^{(1)}_{2,20}) \end{pmatrix} \begin{pmatrix} \delta_{20} \\ \\ \delta_{02} \end{pmatrix}. \tag{6.11}$$

Since Φ_{02} satisfies a hypergeometric equation as well (with the same parameters), we can choose constants $c_{20,i}, i = 1, 2$, s.t.,

$$\left(L(c_{20,1}\,F^{(1)}_{1,20}), L(c_{20,1}\,F^{(1)}_{2,20})\right) = \left(c_{02,1}\,F^{(1)}_{1,02}, c_{02,2}\,F^{(1)}_{2,02}\right).$$

For example, we choose

$$c_{20,1} = \sqrt{2}/8, \quad c_{20,2} = \sqrt{-1}/2, \quad c_{02,1} = \sqrt{2}/8, \quad c_{02,2} = -\sqrt{-1}/2.$$

The mirror map can be chose in the following form: $\Delta_{3,1} = \delta_{20}$ and

$$\begin{pmatrix} \Delta_{1,1} \\ \\ \Delta_{2,1} \end{pmatrix} \bigoplus \begin{pmatrix} \Delta_{1,2} \\ \\ \Delta_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{-1} \\ \\ 1 & -\sqrt{-1} \end{pmatrix} \begin{pmatrix} c_{10}\,\delta_{10} \\ \\ c_{01}\,\delta_{01} \end{pmatrix} \bigoplus \begin{pmatrix} -4\sqrt{-1} & 2\sqrt{-1} \\ \\ -4\sqrt{-1} & -2\sqrt{-1} \end{pmatrix} \begin{pmatrix} c_{11}\,\delta_{11} \\ \\ \delta_{02} \end{pmatrix},$$

where the constants

$$c_{10} = c_{01} = 2^{-1/4}, \quad c_{11} = 1.$$

This identification induces an isomorphism of Frobenius algebras and for the 3-point, degree-1 correlator which is needed in the reconstruction Lemma II.3 we get

$$\begin{aligned} \langle \Delta_{11}, \Delta_{21}, \Delta_{31} \rangle_{0,3} &= \frac{(1-x)^{1/2}\,F^{(1)}_{1,20}\,\pi_A}{8\lambda_W} \\ &= \frac{1}{8}(1-x)^{1/2}\ {}_2F_1\,(1/4, 3/4; 3/2; 1-x)\ {}_2F_1\,(1/4, 1/4; 1; 1-x)\,. \end{aligned}$$

The Fourier series of the correlator has the form $q + 2q^5 + q^9 + 2q^{13} + \cdots$, which in particular implies that the degree-1 part is 1.

Large complex structure limit point for $W_\sigma = X_1^6 + X_2^3 + X_3^2 + \sigma X_1^4 X_2$

We set $x = -\frac{4\sigma^3}{27}$, i.e., $l = 3$ and $C = -4/27$. The j-invariant is

$$j(\sigma) = \frac{4\sigma^3}{27 + 4\sigma^3} = \frac{-x}{1-x}.$$

In this case it is more convenient to construct the mirror map near $\sigma = p_2 = -2^{-2/3}3\eta^2$. We have

$$K = N = 1, \quad \Delta_{02} = 36(1-x)X_1^4 X_2 \pi_A^2.$$

It follows that $\lambda_W = \frac{\sqrt{-3}}{2^{5/6}3\eta^2}$. The mirror map can be constructed as follows:

$$\Delta_{11} = c_{10,2}(1-x)^{1/6}\phi_{10}; \quad \Delta_{15} = c_{31,2}(1-x)^{5/6}\phi_{31};$$

and

$$(\Delta_{21}, \Delta_{12}, \Delta_{31}, \Delta_{13}, \Delta_{22}, \Delta_{14}) = (\delta_{01}, \delta_{20}, \delta_{11}, \delta_{30}, \delta_{21}, \delta_{40}).$$

Here $\delta_{\mathbf{r}}(\mathbf{r} = 01, 20, 11, 30, 21, 40)$ have the same form as in (6.13). After a straightforward computation we get a mirror map that identifies the ring structures by setting

$$\begin{aligned}
\Delta_{11} &= \delta_{10} = -2^{2/3}3^{1/2}(1-x)^{1/6}\,\phi_{10}\,\pi_A,\\
\Delta_{21} &= \delta_{01} = -(-2)^{5/6}3^{1/2}(1-x)^{1/3}\Big(F^{(1)}_{2,20}\,\phi_{01} + (-2)^{-1/3}\,F^{(1)}_{2,01}\,\phi_{20}\Big)\pi_A,\\
\Delta_{12} &= \delta_{20} = (-1)^{1/2}2^{-7/6}3^{1/2}(1-x)^{1/3}\Big(F^{(1)}_{1,20}\,\phi_{01} - 3(-2)^{-1/3}\,F^{(1)}_{1,01}\,\phi_{20}\Big)\pi_A,\\
\Delta_{31} &= \delta_{11} = (-1)^{1/6}2^{2/3}3(1-x)^{1/2}\Big(F^{(1)}_{2,30}\,\phi_{11} + (-2)^{-1/3}\,F^{(1)}_{2,11}\,\phi_{30}\Big)\pi_A,\\
\Delta_{13} &= \delta_{30} = (-2)^{2/3}3^{-1/2}(1-x)^{1/2}\Big(F^{(1)}_{1,20}\,\phi_{11} - 2(-2)^{-1/3}\,F^{(1)}_{1,11}\,\phi_{30}\Big)\pi_A,\\
\Delta_{22} &= \delta_{21} = (-1)^{1/2}2^{3/2}3^{3/2}(1-x)^{2/3}\Big(F^{(1)}_{2,40}\,\phi_{21} + (-2)^{-1/3}\,F^{(1)}_{2,21}\,\phi_{40}\Big)\pi_A,\\
\Delta_{14} &= \delta_{40} = (-1)^{5/6}2^{-5/2}3^{3/2}(1-x)^{2/3}\Big(F^{(1)}_{1,40}\,\phi_{21} - \frac{5}{3}(-2)^{-1/3}\,F^{(1)}_{1,21}\,\phi_{40}\Big)\pi_A,\\
\Delta_{15} &= \delta_{31} = -2^{1/3}3^{1/2}(1-x)^{5/6}\,\phi_{31}\,\pi_A.
\end{aligned}$$

For the 3-point correlator we get

$$\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle_{0,3} = q + 2q^7 + \cdots ,$$

which means that the above identification is a mirror map.

6.2 Mirror to LCSL points at infinity

In this section, we describe the limit behavior of the Saito-Givental theory of ISESs at $\sigma = \infty$. Similar results were already obtained in [MR] for three special choice of an ISES W (with marginal deformation $\phi_{\mathbf{m}} = X_1 X_2 X_3$):

$$X_1^3 + X_2^3 + X_3^3 \in E_6^{(1,1)}, \quad X_1^3 X_2 + X_2^2 X_3 + X_3^2 \in E_7^{(1,1)}, \quad X_1^3 X_2 + X_2^2 + X_3^3 \in E_8^{(1,1)}.$$

Namely, it was proved that the Saito-Givental theory at $\sigma = \infty$ is mirror to the orbifold Gromov-Witten theory respectively of $\mathbb{P}^1_{3,3,3}, \mathbb{P}^1_{4,4,2}$ and $\mathbb{P}^1_{6,3,2}$. It turns out that if W is a Fermat polynomial of type $E_8^{(1,1)}$; then the mirror of the Saito–Givental theory is no longer an orbifold GW theory, but an FJRW theory. This agrees with the physicists' prediction that the monodromy of the Gauss–Manin connection around the large volume limit point should be maximally unipotent, while as we will see below, the monodromy around $\sigma = \infty$ is diagonalizable

The complete answer to the question that what kind of theory is mirror to the Saito–Givental theory at $\sigma = \infty$ for all ISESs is stated in Conjecture I.7. The proof is on a case-by-case basis using the same technique, we compute the initial set of 3- and 4-points genus-0 correlators in the Saito-Givental theory and match them with the corresponding correlators in the orbifold GW or FJRW theories. We will sketch the main steps of the argument on one example and leave the details to the reader. The other cases will appear in a separate paper.

Large complex structure limit point for $W_\sigma = X_1^3X_2 + X_2^2 + X_3^3 + \sigma X_1X_2X_3$

In this example, we substitute $x = -\frac{\sigma^3}{27}$, i.e., $l = 3$ and $C = -1/27$. The j-invariant is

$$j(\sigma) = -\frac{\sigma^3(24+\sigma^3)^3}{27+\sigma^3} = \frac{27x(8-9x)^3}{1-x}.$$

It is convenient to construct the mirror map when σ is near the point $p_1 = -3\eta$. We have

$$K = N = 1, \quad \Delta_{02} = 18(1-x)X_1X_2X_3\pi_A^2.$$

It follows that $\lambda_W = \frac{\sqrt{-3}}{3\eta}$. The Fourier series of $1-x$ in terms of $q = e^{2\pi i\tau/6}$ is

$$1 - x = -27q^6 - 324q^{12} - 2430q^{18} - 13716q^{24} + \cdots \tag{6.12}$$

Note that the largest dimension of a subspace of homogeneous flat sections is 2. Let us assume that $\{\phi_{\mathbf{r}}, \phi_{\mathbf{r}'}\}$ form a basis of the homogeneous subspace of the Jacobi algebra of weight $\deg\phi_{\mathbf{r}} = \deg\phi_{\mathbf{r}'}$. The period integral $\Phi_{\mathbf{r}}(\sigma)$ is a solution to a hypergeometric equation with weights $\alpha_{\mathbf{r}}, \beta_{\mathbf{r}}, \gamma_{\mathbf{r}}$. We choose a new basis of sections with polynomials defined by

$$(\delta_{\mathbf{r}}, \delta_{\mathbf{r}'}) = (\phi_{\mathbf{r}}\,\pi_A, \phi_{\mathbf{r}'}\,\pi_A)\begin{pmatrix} c_{\mathbf{r},1}F^{(1)}_{1,\mathbf{r}}(x) & c_{\mathbf{r}',1}F^{(1)}_{1,\mathbf{r}'}(x) \\ c_{\mathbf{r},2}F^{(1)}_{2,\mathbf{r}}(x) & c_{\mathbf{r}',2}F^{(1)}_{2,\mathbf{r}'}(x) \end{pmatrix}^{-1}, \tag{6.13}$$

where $c_{\mathbf{r},i}, c_{\mathbf{r}',i} (i = 1, 2)$ are some constants and

$$\left\{F^{(1)}_{1,\mathbf{r}}(x) := F^{(1)}_1(x), F^{(1)}_{2,\mathbf{r}}(x) := F^{(1)}_2(x)\right\}$$

is the basis of solutions (4.31) to the hypergeometric equation with weights $(\alpha_{\mathbf{r}}, \beta_{\mathbf{r}}, \gamma_{\mathbf{r}})$. If $\Phi_{\mathbf{r}}$ satisfies a first order differential equation, we set

$$F^{(1)}_1(x) = 0, \quad F^{(1)}_2(x) = {}_1F_0(\alpha_{\mathbf{r}}; 1-x)$$

and use the same formula in order to define $\delta_{\mathbf{r}}$. We choose $c_{\mathbf{r},i}$ such that

Table 6.1:

$\mathbf{r} =$	100	200	001	101	010	201	110	011
$c_{\mathbf{r},1} =$	0	$\frac{\sqrt{-3}\eta}{3}$	0	$\frac{2\sqrt{-1}\eta}{9}$	$-\frac{\sqrt{-1}\eta^2}{9}$	$\frac{\sqrt{-1}\eta^2}{3}$	0	0
$c_{\mathbf{r},2} =$	$\sqrt{3}$	$-\frac{\sqrt{-1}}{3}$	$-\frac{\sqrt{-1}}{3}$	$-\frac{\sqrt{3}\eta}{3}$	$-\frac{\sqrt{3}\eta^2}{3}$	$\frac{\sqrt{-3}}{3}$	$\frac{\sqrt{-3}}{3}$	$\sqrt{3}$

Now we can construct a map as follows:

$$\Delta_{11} = c_{100,2}(1-x)^{1/6}\Phi_{100}; \quad \Delta_{15} = c_{011,2}(1-x)^{5/6}\Phi_{011};$$

and

$$(\Delta_{21}, \Delta_{12}, \Delta_{31}, \Delta_{13}, \Delta_{22}, \Delta_{14}) = (\delta_{200}, \delta_{001}, \delta_{101}, \delta_{010}, \delta_{201}, \delta_{110}) .$$

This is actually a mirror map. It preserves the Chen-Ruan product and

$$\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle_{0,3} = \frac{1}{\sqrt{3}}(1-x)^{1/6}\ {}_2F_1\,(1/3, -1/6; 1/2; 1-x)\,\pi_A = q + 2q^7 + \cdots .$$

6.3 Mirror to other FJRW-points

The case of Fermat type $E_8^{(1,1)}$

Let $W(\sigma) := X_1^6 + X_2^3 + X_3^2 + \sigma X_1^4 X_2$. As usual we substitute $x = -\frac{4}{27}\sigma^3$. The j-invariant is

$$j(\sigma) = \frac{4\sigma^3}{27 + 4\sigma^3} = \frac{-x}{1-x}.$$

The Picard-Fuchs equation for the periods π_A has weights $(\alpha, \beta, \gamma) = (1/12, 7/12, 2/3)$. Since $\alpha - \beta$ is not an integer, the monodromy is diagonalizable and we have the following basis of solutions (eigenvectors for the monodromy around $\sigma = \infty$) near $x = \infty$:

$$\pi_{A_\infty} := x^{-1/12}\ {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; \frac{1}{2}; x^{-1}\Big), \quad \pi_{B_\infty} := \lambda_W\ x^{-7/12}\ {}_2F_1\Big(\frac{7}{12}, \frac{11}{12}; \frac{3}{2}; x^{-1}\Big),$$

where the constant λ_W will be fixed later on. Put

$$t_{-1} = \frac{\pi_{B_\infty}}{\pi_{A_\infty}} \approx \lambda_W\ x^{-\frac{1}{2}},$$

where $\approx$ means that we truncated terms of order $O(\sigma^2)$. It is easy to check (by using the differential equation for the periods) that when restricted to the subspace of marginal deformation, t_{-1} is a degree 0 flat coordinate, i.e., the residue pairing $\langle 1, \partial/\partial t_{-1}\rangle$ is a constant.

To construct the mirror map for non-integral degrees, we have to find a basis of homogeneous flat sections (with non-integer degrees) of the Gauss–Manin connection near $\sigma = \infty$. The periods corresponding to the polynomials ϕ_{01} and ϕ_{20} satisfy

$$\begin{cases} (27+4\sigma^3)\partial_\sigma \Phi_{20} = -3\sigma^2\,\Phi_{20} - (9/2)\Phi_{01} \\ (27+4\sigma^3)\partial_\sigma \Phi_{01} = -\sigma^2\,\Phi_{01} + (9\sigma/2)\Phi_{20} \end{cases}$$

Moreover, as we already know Φ_{01} satisfies a hypergeometric equation. The corresponding weights are $(\alpha_{01}, \beta_{01}, \gamma_{01}) = (1/12, 7/12, 1/3)$. Let

$$L_{20} := \frac{2}{9\sigma}\Big((27+4\sigma^4)\partial_\sigma \Phi_{01} + \sigma^2\Phi_{01}\Big).$$

Then the solutions to the above system have the form

$$\begin{pmatrix} \Phi_{01} \\ \\ \Phi_{20} \end{pmatrix} = \begin{pmatrix} F^{(\infty)}_{1,01}(x) & F^{(\infty)}_{2,01}(x) \\ \\ L_{20}F^{(\infty)}_{1,01}(x) & L_{20}F^{(\infty)}_{2,01}(x) \end{pmatrix} \begin{pmatrix} A_{01} \\ \\ A_{20} \end{pmatrix}$$

Solving for A_{01} and A_{20} we obtain two flat sections of degree $\frac{1}{3}$. The remaining flat sections can be found in a similar way. We get

$$A_{10} = (x-1)^{1/6}\Phi_{10}, \quad A_{31} = (x-1)^{5/6}\Phi_{31}$$

and

$$\begin{pmatrix} \Phi_{k,1} \\ \\ \Phi_{k+2,0} \end{pmatrix} = \begin{pmatrix} F^{(\infty)}_{1,(k,1)}(x) & F^{(\infty)}_{2,(k,1)}(x) \\ \\ L_{k+2,0}F^{(\infty)}_{1,(k,1)}(x) & L_{k+2,0}F^{(\infty)}_{2,(k,1)}(x) \end{pmatrix} \begin{pmatrix} A_{k,1} \\ \\ A_{k+2,0} \end{pmatrix}, \quad k = 0,1,2$$

where

$$L_{30} = \frac{1}{6\sigma}\Big((27+4\sigma^4)\partial_\sigma\Phi_{11} + 2\sigma^2\Phi_{11}\Big), \quad L_{40} = \frac{2}{15\sigma}\Big((27+4\sigma^4)\partial_\sigma\Phi_{21} + 3\sigma^2\Phi_{21}\Big).$$

Let $\delta_{\mathbf{r}}(\mathbf{s},\mathbf{X})$ be polynomials, such that the geometric sections $[\delta_{\mathbf{r}}\omega] = c_{\mathbf{r}}A_{\mathbf{r}}$, see (5.2). Here $c_{\mathbf{r}}$ are given in the table below

$\mathbf{r} =$	10	01	20	11	30	21	40	31
$c_{\mathbf{r}}$	λ_1	λ_2	$-\frac{\lambda_1^2 C_0}{3}$	$\lambda_1\lambda_2$	$-\lambda_1^3 C_0$	$\lambda_1\lambda_2^2$	$\frac{4\lambda_1^4 C_0}{5}$	$\frac{2\lambda_1^5 C_0}{9}$

The constants appearing in the table are given as follows:

$$\lambda_1^6 = 24C_0^2, \quad \lambda_2^2 = \frac{\lambda_1^4}{C_0}, \quad C_0^3 = -\frac{27}{4}. \tag{6.14}$$

Now we compute the pairing and the necessary genus-0 correlators. The pairing is

$$\langle\delta_{10},\delta_{31}\rangle = \langle\delta_{01},\delta_{21}\rangle = \langle\delta_{20},\delta_{40}\rangle = \langle\delta_{11},\delta_{11}\rangle = \langle\delta_{30},\delta_{30}\rangle = 1.$$

All 3-point correlator functions that do not have insertion 1 (otherwise the correlator reduces to a 2-point one) have a limit at $\sigma = \infty$. The non-zero limits are as follows:

$$\langle\delta_{10},\delta_{10},\delta_{40}\rangle_{0,3} = \langle\delta_{10},\delta_{01},\delta_{11}\rangle_{0,3} = \langle\delta_{01},\delta_{01},\delta_{20}\rangle_{0,3} = 1.$$

$$\langle\delta_{10},\delta_{20},\delta_{30}\rangle_{0,3} = \langle\delta_{20},\delta_{20},\delta_{20}\rangle_{0,3} = -3.$$

In other words, the Jacobi algebra extends over $\sigma = \infty$. If we denote the extension by $\mathcal{Q}_{W^\infty}$, then it is not hard to see that δ_{10} and δ_{01} are generators and we have

$$\mathcal{Q}_{W^\infty} := \mathbb{C}[\delta_{10},\delta_{01}]/\Big(4\delta_{10}^3\delta_{01}, \delta_{10}^4 + 3\delta_{01}^2\Big).$$

Finally, the nonzero 4-point genus-0 basic correlators are

$$\langle\delta_{01},\delta_{01},\delta_{01},\delta_{-1}\rangle_{0,4} = \frac{\partial}{\partial t_{-1}}\langle\lambda_2^3(x^{1/12}\Phi_{01} + \frac{3}{4C_0}x^{-1/4}\Phi_{20})^3\rangle\Big|_{x=\infty} =$$
$$\lambda_2^3\frac{\partial}{\partial t_{-1}}\left(x^{1/4}\langle\Phi_{03}\rangle + \frac{9}{4C_0}x^{7/12}\langle\Phi_{22}\rangle\right) = -\frac{\lambda_2^2 C_0}{4\lambda_1^4}\frac{\partial}{\partial t_{-1}}\left(\lambda_W\, x^{-1/2}\right) = -\frac{1}{4}.$$

and

$$\langle \delta_{10}, \delta_{10}, \delta_{10}^2\delta_{01}, \delta_{-1}\rangle_{0,4} = -\frac{1}{4}.$$

where in order to achieve the above identities we set

$$\lambda_W = \frac{\lambda_1^4\lambda_2}{54}.$$

Recall that ρ_1, ρ_2 are generators of the FJRW ring corresponding to the pair $\left(W' = X_1^4X_2 + X_2^3 + X_3^2, G_{W'}\right)$. Using the reconstruction lemma in FJRW theory (see Lemma III.10, it is easy to check that the map

$$\left(\rho_1, \rho_2\right) \mapsto \left((-1)^{5/6}\delta_{01}, (-1)^{2/3}\delta_{10}\right).$$

is a mirror symmetry map, i.e., it induces an isomorphism between the FJRW theory of $(W', G_{W'})$ and the Saito-Givental limit of $W = X_1^6 + X_2^3 + X_3^2 + \sigma X_1^4X_2$ at $\sigma = \infty$.

CHAPTER VII

Landau-Ginzburg/Calabi-Yau correspondence

7.1 Givental's quantization formula

Following Givental, we introduce the vector space $\mathcal{H} = H((z))$ of formal Laurent series in z^{-1}. Furthermore, $\mathcal{H}$ is equipped with the following symplectic structure Ω:

$$\Omega(f(z), g(z)) = \mathrm{res}_{z=0}(f(-z), g(z))dz, \quad f(z), g(z) \in \mathcal{H},$$

where for brevity we put $(a, b) = \eta(a, b)$ for $a, b \in H$. Note that

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

with $\mathcal{H}_+ = H[z]$ and $\mathcal{H}_- = z^{-1}H[[z^{-1}]]$, which allows us to identify $\mathcal{H}$ with $T^*\mathcal{H}_+$. We fix a Darboux coordinate system $q_k^i, p_{l,j}$ for $\mathcal{H}$ via

$$f(z) = \sum_{k=0}^{\infty}\sum_{i=0}^{N-1} q_k^i\,\partial_i z^k + \sum_{l=0}^{\infty}\sum_{j=0}^{N-1} p_{l,j}\,\partial^j(-z)^{-l-1} \in \mathcal{H},$$

For convenience, we put

$$\mathbf{q}_k := (q_k^1, \cdots, q_k^N) \qquad \text{and} \qquad \mathbf{q} := (\mathbf{q}_0, \mathbf{q}_1, \cdots). \tag{7.1}$$

In this paper, we focus on the subgroup $\mathcal{L}^{(2)}\mathrm{GL}(H)$ of the loop group $\mathcal{L}\mathrm{GL}(H)$ consisting of symplectomorphisms $T : \mathcal{H} \to \mathcal{H}$. Note that such symplectomorphisms are defined by the following equation:

$${}^*T(-z)T(z) = \mathrm{Id},$$

where ${}^*T$ is the adjoint operator with respect to the bi-linear pairing η, i.e.,

$$({}^*Tf, g) = (f, Tg).$$

We will allow symplectomorphism E of the following form:

$$E := \mathrm{Id} + E_1 z + E_2 z^2 + \cdots \in \mathrm{End}(H)[[z]].$$

They form a group which we denote by $\mathcal{L}_+^{(2)}\mathrm{GL}(H)$ and we refer to its elements as *upper-triangular* transformations.

Next, we want to define the quantization $\widehat{E}$. Note that $A = \log E$ is a well-defined *infinitesimal symplectomorphism*, i.e., ${}^*A = -A$. For any infinitesimal symplectomorphism A, we can associate a *quadratic Hamiltonian* h_A on $\mathcal{H}$,

$$h_A(f) = \frac{1}{2}\Omega(Af, f). \tag{7.2}$$

The quadratic Hamiltonians are quantized by the rules:

$$(p_{k,i}p_{l,j})\widehat{\ } = \hbar\frac{\partial^2}{\partial q_k^i \partial q_l^j}, \quad (p_{k,i}q_l^j)\widehat{\ } = (q_l^j p_{k,i})\widehat{\ } = q_l^j \frac{\partial}{\partial q_k^i}, \quad (q_k^i q_l^j)\widehat{\ } = \frac{q_k^i q_l^j}{\hbar}. \tag{7.3}$$

The quantization of E is defined by

$$\widehat{E} = e^{\widehat{A}} := e^{\widehat{h_A}}.$$

For an upper-triangular symplectomorphism E, there is an explicit formula for the quantization $\widehat{E}$. Put

$$\mathbf{q}(z) = \sum_{k=0}^{\infty}\sum_{i=0}^{N-1} q_k^i\, \partial_i z^k \in H[[z]].$$

Denote the *dilaton shift* by $\widetilde{\mathbf{q}}(z) = \mathbf{q}(z) + \mathbf{1}z$, i.e., $\widetilde{q}_k^i = q_k^i + \delta_k^1\delta_0^i$. Recall that *the ancestor GW potential* of X is

$$\mathcal{A}^X(\hbar, \mathbf{q}(z)) := \exp\Big(\sum_{g,n}\sum_{\beta\in\mathrm{NE}(X)}\sum_{\iota_i,k_i=0}^{\infty} \frac{\hbar^{g-1}\langle \tau_{\iota_1}\partial_{k_1}, \cdots, \tau_{\iota_n}\partial_{k_n}\rangle_{g,n,\beta}^X q^\beta}{n!}\prod_{i=1}^n \widetilde{q}_{k_i}^{\iota_i}\Big). \tag{7.4}$$

$\mathcal{A}^X(\hbar, \mathbf{q}(z))$ belongs to a Fock space $\mathbb{C}[[\mathbf{q}_0, \widetilde{\mathbf{q}}_1, \mathbf{q}_2, \cdots]]$. The action of the quantization operator $\widehat{E}$, whenever it makes sense, is given by the following formula:

$$\widehat{E}(\mathcal{A}^X(\hbar, \mathbf{q}(z))) = \left(e^{W_E}\mathcal{A}^X(\hbar, \mathbf{q}(z))\right)\Big|_{\mathbf{q}\mapsto E^{-1}\mathbf{q}}, \tag{7.5}$$

where $E^{-1}\mathbf{q}$ is the change of $\mathbf{q}$-coordinate

$$(E^{-1}\mathbf{q})^i_k = \sum_{l=0}^{k}\sum_{j=0}^{N-1}(E^{-1})^{ji}_l q^j_{k-l}.$$

And W_E is the quadratic differential operator

$$W_E := \frac{\hbar}{2}\sum_{k,l=0}^{\infty}\sum_{i,j=0}^{N-1}\left(\partial^i, V_{kl}(\partial^j)\right)\frac{\partial^2}{\partial q^i_k \partial q^j_l}, \tag{7.6}$$

whose coefficients $V_{kl} \in \mathrm{End}(H)$ are given by

$$\sum_{k,l\geq 0} V_{kl}(-z)^k(-w)^l = \frac{E^*(z)E(w) - \mathrm{Id}}{z+w}. \tag{7.7}$$

Remark VII.1. Givental also considered the quantization of a general symplectomorphism of the form e^A. For example, A could be lower triangular in the sense containing the negative power of z. The lower triangular one can not be lift to cycle level. Hence, it will not be considered here.

7.1.1 A quantization operator in singularity theory

Suppose that W is one of the three families of simple elliptic singularities under consideration. Recall the global Frobenius manifold structures on $\mathcal{M}$. First we recall the definiton of Givental's quantization operator and then we use it to define a CohFT $\Lambda^W(\mathbf{t})$ over the semisimple loci $\mathcal{M}_{ss}$.

Let $\mathcal{K} \subset \mathcal{M}$ be the set of points $\mathbf{t}$ such that $u_i(\mathbf{s}(\mathbf{t})) = u_j(\mathbf{s}(\mathbf{t}))$ for some $i \neq j$. We call this set the *caustic* and put $\mathcal{M}_{ss}$ for its complement. Note that the points $\mathbf{t} \in \mathcal{M}_{ss}$ are semisimple, i.e., the critical values $u_i(\mathbf{s}(\mathbf{t}))$ $(1 \leq i \leq \mu)$ form a coordinate system locally near $\mathbf{t}$. Let $\mathbf{t} \in \mathcal{M}_{ss}$; then we have an isomorphism

$$\Psi(\mathbf{t}) : \mathbb{C}^{\mu} \to T_{\mathbf{t}}\mathcal{M}, \quad e_i \mapsto \sqrt{\Delta_i(\mathbf{s}(\mathbf{t}))}\,\frac{\partial}{\partial u_i(\mathbf{s}(\mathbf{t}))},$$

where $\Delta_i(\mathbf{s}(\mathbf{t}))$ is defined by

$$\Big(\frac{\partial}{\partial u_i(\mathbf{s}(\mathbf{t}))}, \frac{\partial}{\partial u_j(\mathbf{s}(\mathbf{t}))}\Big) = \frac{\delta_{ij}}{\Delta_i(\mathbf{s}(\mathbf{t}))},$$

and we identify $T_{\mathbf{t}}\mathcal{M}$ with H via the flat metric, i.e.,

$$\frac{\partial}{\partial u_i} = \sum_{j=0}^{\mu-1} \frac{\partial t_j}{\partial u_i}\,\partial_j, \quad 1 \le i \le \mu.$$

$\Psi_{\mathbf{t}}$ diagonalizes the Frobenius multiplication and the residue pairing:

$$e_i \bullet e_j = \delta_{i,j}\sqrt{\Delta_i(\mathbf{s}(\mathbf{t}))}\,e_i, \quad (e_i, e_j) = \delta_{ij}.$$

The system of differential equations (4.8) and (4.9) admits a unique formal solution of the type

$$\Psi(\mathbf{t})R(\mathbf{t})\,e^{U(\mathbf{t})/z}, \quad R(\mathbf{t}) = \mathrm{Id} + \sum_{k=1}^{\infty} R_k(\mathbf{t})z^k \in \mathrm{End}(\mathbb{C}^{\mu})[[z]].$$

where $U(\mathbf{t})$ is a diagonal matrix with entries $u_1(\mathbf{s}(\mathbf{t})), \ldots, u_{\mu}(\mathbf{s}(\mathbf{t}))$ on the diagonal, cf.[Du, Gi2]. As we work for isolated singularities, the homogeneity condition implies $R_k(\mathbf{s})$ are uniquely solved from $R_0(\mathbf{s})$ by inductively.

Following Givental, the *formal* ancestor potential

$$\mathcal{A}^{formal}(\mathbf{t})(\hbar, \mathbf{q}(z))$$

of (the germ of) the Frobenius structure $(H, \eta, \bigstar_{\mathbf{t}})$ is defined by

$$\widehat{\Psi}(\mathbf{t})\,\widehat{R}(\mathbf{t})\,e^{(U(\mathbf{t})/z)\hat{}}\,\prod_{i=1}^{\mu} \mathcal{A}^{\mathrm{pt}}(\hbar\Delta_i(\mathbf{t}), {}^{i}\widetilde{\mathbf{q}}(z)\,\sqrt{\Delta_i(\mathbf{t})}), \tag{7.8}$$

where $\widehat{\Psi}$ means change of the variables $\mathbf{q}(z) \mapsto \Psi^{-1}(\mathbf{t})\,\mathbf{q}(z)$ and $\mathcal{A}^{\mathrm{pt}}$ is the total ancestor potential of the CohFT $I^{N=1,\Delta=1}$.

7.2 Symplectic transformation and analytic continuation

As we recall for the construction of B-model total ancestor potential, it depends on the choices of a flat basis of the state space. And the choice of a flat basis is essentially coming from a choice of symplectic basis in $H_1(E_\sigma, \mathbb{C})$. For two special limits σ_1 and σ_2, let us denote the choice of symlectic basis by $(A_i, B_i) \in H_1(E_{\sigma_i}, \mathbb{C})$ for $i = 1, 2$. We can analyticlally continuate the basis (A_1, B_1) to σ_2 by a path C_{σ_1,σ_2} connecting from σ_1 to σ_2. We denote the new basis at σ_2 again by (A_1, B_1). The difference between two basis can be captured by a transformation, U_{σ_1,σ_2}, where

$$U_{\sigma_1,\sigma_2}\begin{pmatrix} A_1 \\ \\ B_1 \end{pmatrix} = \begin{pmatrix} A_2 \\ \\ B_2 \end{pmatrix}.$$

This transformation gives a symplectic transformation on space $\mathcal{H}$. We denote it again by U_{σ_1,σ_2}. According to [MR], the difference of two total ancestor potentials at σ_1 and σ_2 are related by the quantization operator for this symplectic transformation, up to analytic transformation. Combine all the mirror theorems we had in previous chapters, we obtain a proof of Theorem I.2.

CHAPTER VIII

Modularity

8.1 Cycle-valued quantization

In this chapter, we will discuss modularity of Gromov-Witten theory in a cycle-valued version. For reader's convenience, we introduce a cycle-valued version of Givental's quantization formula here. The key observation is due to Teleman. Teleman [Te] was able to lift the quantization of an upper triangular symplectic transformation to the level of cohomological field theory. Let us describe his construction. According to formula (7.5), the action of $\widehat{E}$ is a composition of two operations: exponential of the Laplace type operator (7.6) followed by the *coordinate change* $\mathbf{q} \mapsto E^{-1}\mathbf{q}$.

Coordinate Change

Let $\Lambda_{g,n}$ be any linear function on $H^{\otimes n}$ with values in the cohomology ring of $\overline{\mathcal{M}}_{g,n}$. We can extend $\Lambda_{g,n}$ from $H^{\otimes n}$ to $\mathcal{H}_+^{\otimes n}$ uniquely so that multiplication by z is compatible with the multiplication by psi-classes, i.e.,,

$$\Lambda_{g,n}(\sum_{i\geq 0}\gamma_1 z^i, \cdots) = \sum_{i\geq 0}\Lambda_{g,n}(\gamma_1, \cdots)\psi_1^i. \tag{8.1}$$

Given an isomorphis of $\mathbb{C}[z]$-modules

$$\Phi(z) : H_1[[z]] \to H_2[[z]],$$

we define

$$(\Phi(z)\circ\Lambda)_{g,n}(\gamma_1,\cdots,\gamma_n)=\Lambda_{g,n}(\Phi(z)^{-1}(\gamma_1),\cdots,\Phi(z)^{-1}(\gamma_n))\in H^*(\overline{\mathcal{M}}_{g,n},\mathbb{C}).$$

Note that even if Λ is a CohFT, $\Phi(z)\circ\Lambda$ might fail to be a CohFT.

Feynman type sum

The action of the exponential of the Laplacian (7.6) can be described in terms of sum over graphs. Let us explain this in some more details. For a given graph Γ let us denote by $V(\Gamma)$ the set of vertices, $E(\Gamma)$ the set of edges, and by $T(V)$ the set of tails. For a fixed vertex $v\in V(\Gamma)$ we denote by $E_v(\Gamma)$ and $T_v(\Gamma)$ respectively the set of edges and tails incident with v. The graph is *decorated* in the following way: each vertex v is assigned a non-negative number g_v called genus of v; there is a bijection $t\mapsto m(t)$ between the set of tails and the set of integers $\{1,2,\ldots,\mathrm{Card}(T(\Gamma))\}$, and finally every flag (v,e) (i.e., a pair consists of a vertex and an incident edge) is decorated with a vector $z^k\partial^i$ ($k\geq 0$).

Furthermore, for a given edge e we define a *propagator* V_e as follows. Let v', v'' be the two vertexes incident with e and let $z^{k'}\partial^{i'}$ and $z^{k''}\partial^{i''}$ be the labels respectively of the flags (v',e) and (v'',e); then we define

$$V_e=\left(\partial^{i'},V_{k',k''}\partial^{i''}\right).$$

Note that since ${}^*V_{k',k''}=V_{k'',k'}$ the definition of V_e is independent of the orientation of the edge e. For every vertex v we define the differential operator

$$D^v_{\mathbf{q}}=\prod_{e\in E_v(\Gamma)}\partial/\partial q^{i(e)}_{k(e)},$$

where $z^{k(e)}\partial^{(i(e)}$ is the label of the flag (v,e).

Given any formal function $\mathcal{A}(\hbar;\mathbf{q})=\exp\Big(\sum\hbar^{g-1}\mathcal{F}^{(g)}(\mathbf{q})\Big)$, we have

$$e^{W_E}\,\mathcal{A}(\hbar;\mathbf{q})=\exp\Big(\sum_{\Gamma}\frac{1}{|\mathrm{Aut}(\Gamma)|}\prod_{e\in E(\Gamma)}V_e\prod_{v\in V(\Gamma)}D^v_{\mathbf{q}}\mathcal{F}^{(g_v)}(\mathbf{q})\Big),\tag{8.2}$$

where the sum is over all connected decorated graphs Γ and $|\mathrm{Aut}(\Gamma)|$ is the number of automorphisms of Γ compatible with the decoration.

Motivated by formula (8.2) we define

$$(e^{W_E} \circ \Lambda)_{g,n}(\gamma_1 \otimes \cdots \otimes \gamma_n) \tag{8.3}$$

by the following formula:

$$\sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{e \in E(\Gamma)} V_e \prod_{v \in V(\Gamma)} \Lambda_{g_v, r_v + n_v}\Big(\otimes_{e \in E_v(\Gamma)} \partial_{i(e)} \psi^{k(e)} \otimes_{t \in T_v(\Gamma)} \gamma_{m(t)} \Big),$$

where $r_v = \mathrm{Card}(E_v(\Gamma))$, $n_v = \mathrm{Card}(T_v(\Gamma))$, and the sum is over all connected, decorated, genus-g graphs Γ with n tails. Note that this definition is compatible with (8.2) in a sense that the potential of the multi-linear maps (8.3) coincides with (8.2).

For an upper-triangular symplectic transformation E, we define

$$\widehat{E} \circ \Lambda := E \circ (e^{W_E} \circ \Lambda). \tag{8.4}$$

Using induction on the number of nodes, it is not hard to check that $\widehat{E} \circ \Lambda$ is a CohFT (see [Te]).

Classification of semi-simple CohFT

Let $(H, \eta, \bullet)$ be a semi-simple Frobenius algebra. We pick an orthonormal basis $\{e_i\}$ of H, which allows us to identify $(H, \eta, \bullet)$ with the Frobenius algebra of a trivial CohFT, i.e.,, the state space of $I^{N,\Delta}$ for a particular Δ (see (2.2)). In this section we would like to recall the classification of all CohFTs whose state space is $(H, \eta, \bullet)$. According to Teleman (see [Te], Theorem 1) we have the following higher-genus reconstruction result.

Proposition VIII.1. *([Te]) If Λ is a homogeneous CohFT with its underlying Frobenius algebra $(H, \eta, \bullet)$. Assume H has a flat identity and $\bigstar_{\mathbf{t}}$ is (formal) semi-simple; then*

$${}_{\mathbf{t}}\Lambda = \widehat{\Psi}(\mathbf{t}) \circ (T_z \circ \widehat{R}(\mathbf{t}) \circ T_z^{-1}) \circ I^{N,\Delta(\mathbf{t})},$$

where $T_z := T_{\mathbf{1}z}$ *is a translation defined by*

$$(T_{\mathbf{1}z} \circ \Lambda)_{g,n}(\gamma_1, \cdots, \gamma_n) := \sum_{k=0}^{\infty} \frac{1}{k!} \pi_* \Lambda_{g,n+k}(\gamma_1, \cdots, \gamma_n, \psi_{n+1}, \cdots, \psi_{n+k})$$

8.2 Cycle-valued version of B-model potential

Givental used $R(\mathbf{t})$ to define a higher genus generating function over $\mathcal{M}_{ss}$. In this chapter, we would like to enhance his definition to Cohomological Field Theory.

For any semisimple point $\mathbf{t} \in \mathcal{M}_{ss}$, we define a CohFT with a flat identity and a state space H,

$$\Lambda^W(\mathbf{t}) := \Psi(\mathbf{t}) \circ T_z \circ \widehat{R}(\mathbf{t}) \circ T_z^{-1} \circ I^{\mu,\Delta(\mathbf{t})}. \tag{8.5}$$

We are interested in the loci of points $\mathbf{t} = (t, 0) \in \mathbb{H} \times \mathbb{C}^{\mu-1}$, which are never semisimple. To continue our B-model discussion, we need to prove that $\Lambda^W(\mathbf{t})$ extends holomorphically for all $\mathbf{t} \in \mathcal{M}$. To begin with, let us fix g, n, and $\gamma_i \in H$; for convenience, we denote by

$$\Lambda^W_{g,n}(\mathbf{t}) := (\Lambda^W(\mathbf{t}))_{g,n}.$$

$\Lambda^W_{g,n}(\mathbf{t})(\gamma_1, \ldots, \gamma_n)$ is a linear combination of cohomology classes on $\overline{\mathcal{M}}_{g,n}$ whose coefficients are functions on $\mathcal{M}$.

Lemma VIII.2. *The coefficients of* $\Lambda^W_{g,n}(\mathbf{t})(\gamma_1, \ldots, \gamma_n)$ *are meromorphic functions on* $\mathcal{M}$ *with at most finite order poles along the caustic* $\mathcal{K}$.

Proof. By definition, the CohFT (8.5) depends only on the choice of a canonical coordinate system $u(\mathbf{t}) := (u_1(\mathbf{s}(\mathbf{t}), \ldots, u_\mu(\mathbf{s}(\mathbf{t}))$. The latter is uniquely determined up to permutation. Note that (8.5) is permutation-invariant, i.e., it does not matter how we order the canonical coordinates. On the other hand, up to a permutation $u(\mathbf{t})$ is invariant under the analytical continuation along a closed loop in $\mathcal{M}_{ss}$. It follows that $\Lambda^W_{g,n}(\mathbf{t})$ is a single valued function on $\mathcal{M}_{ss}$.

We need only to prove that the poles along $\mathcal{K}$ have finite order. Note that according to the definition of the class (8.3) only finitely many graphs Γ contribute. The reason for this is that in order to have a non-zero contribution, we must have

$$\sum_{e\in E_v(\Gamma)} k(e) \ \leq\ 3g_v - 3 + r_v + n_v.$$

Summing up these inequalities, we get

$$\sum_v \sum_{e\in E_v(\Gamma)} k(e) \ \leq 3(g-1) - 3\mathrm{Card}(E(\Gamma)) + \sum_v r_v + n,$$

However

$$\sum_v r_v = 2\mathrm{Card}(E(\Gamma)),$$

which implies that the number of edges of Γ is bounded by $3g-3+n$. This proves that there are finitely many possibilities for Γ. Moreover, there are only finitely many possibilities for $k(e)$, i.e., our class is a rational function on the entries of only finitely many R_k. Since each R_k has only a finite order pole along the caustic the Lemma follows. □

We will prove below that $\Lambda^W_{g,n}(\mathbf{t})$ is convergent near the point $(\sqrt{-1}\,\infty, 0) \in \overline{\mathbb{H}}\times\mathbb{C}^{\mu-1}$ and that it extends holomorphically through the caustic (see Theorem VIII.10 and Proposition VIII.12). Thus $\Lambda^W(\mathbf{t})$ is a CohFT for all $\mathbf{t} \in \mathcal{M}$. In particular,

(8.6) $$\Lambda^W_{g,n}(t) = \lim_{\mathbf{t}\in\mathcal{M}_{ss}\to(t,0)} \Lambda^W_{g,n}(\mathbf{t})$$

for all $t \in \mathbb{H} = \mathbb{H}\times\{0\} \subset \mathcal{M}$.

8.3 Modular transformation

Using the residue pairing we identify $T^*\mathcal{M}$ and $T\mathcal{M}$, i.e., $dt_i = \partial_{i'}$. We also identify $\mathrm{End}(H)$ with the space of $\mu\times\mu$ matrices via $A \mapsto (A_{ij})$, where the entries A_{ij} are defined in the standard way, i.e.,

$$A(dt_j) = \sum_{i=-1}^{\mu-2} A_{ij} dt_i\,.$$

Recall the notation from Chapter IV: a loop C in Σ, inducing via the Gauss-Manin connection a monodromy transformation ν on vanishing homology and a transformation of the flat coordinates via analytic continuation $\mathbf{t} \mapsto \nu(\mathbf{t})$. The latter induces a monodromy transformation of the stationary phase asymptotics, which was computed in [MR], Lemma 4.1. When $W = X_1^3 + X_2^3 + X_3^3$, let

$$(8.7) \qquad M_\nu(\mathbf{t}) = \begin{pmatrix} j_\nu(t)^{-1} & * & * & * \\ 0 & j_\nu(t) & 0 & 0 \\ 0 & * & e^{4\pi i k/3} I_3 & 0 \\ 0 & * & 0 & e^{2\pi i k/3} I_3 \end{pmatrix} \in \mathrm{End}(H)[[z]].$$

where

$$M_{-1,j} = -e^{2\pi i d_j k} n_{12}\, j_\nu^{-1}(t)\, t_j, \quad 1 \le j \le 6$$

and

$$M_{-1,0} = -n_{12} z - \frac{n_{12}^2}{2 j_\nu(t)} \sum_{i=1}^{6} t_i t_{i'}, \quad M_{i,0} = n_{12} t_{i'}, \quad 1 \le i \le 6.$$

Lemma VIII.3. *[MR] The analytic continuation along C transforms*

$$\Psi(\mathbf{t}) R(\mathbf{t}) e^{U(\mathbf{t})/z} \quad \textit{into} \quad {}^{\mathrm{T}}M_\nu(\mathbf{t}) \Psi_{\mathbf{t}} R(\mathbf{t}) e^{U(\mathbf{t})/z} P,$$

where P is a permutation matrix and ${}^{\mathrm{T}}$ means transposition. □

The CohFT constructed by the analytical continuation along C of $\Lambda^W(\mathbf{t})$ will be denoted by

$$\Lambda^W_{g,n}(\nu(\mathbf{t})) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C}) \otimes (H^\vee)^{\otimes n}.$$

Restricting to $\mathbf{t}_{\ge 0} = 0$, we have

$${}^{\mathrm{T}}M_\nu(t) := \lim_{\mathbf{t}_{\ge 0} \to 0} {}^{\mathrm{T}}M_\nu(\mathbf{t}) = j_\nu^{-1}(t)\, J_\nu(t).$$

With

$$J_\nu(t) := \begin{pmatrix} 1 & 0 \\ 0 & j_\nu^2(t) \end{pmatrix} \bigoplus j_\nu(t)\, e^{4\pi i\, k/3}\, I_3 \bigoplus j_\nu(t)\, e^{2\pi i\, k/3}\, I_3 \in \mathrm{End}(H)[[z]]. \tag{8.8}$$

Now let

$$X_{\nu,t}(z) = \begin{pmatrix} 1 & -n_{12} z / j_\nu(t) \\ 0 & 1 \end{pmatrix} \bigoplus I_6 \in \mathrm{End}(H)[[z]]. \tag{8.9}$$

Theorem VIII.4. *The analytic continuation transforms the Coh FT as follows:*

$$\Lambda^W(\nu(t)) = J_\nu^{-1}(t) \circ \widehat{X}_{\nu,t}(z) \circ \Lambda^W(t). \tag{8.10}$$

Proof. The calculation in [MR] also works on cycle-valued level. □

Now we give a lemma which is very useful later on.

Lemma VIII.5. *Let $E(z) \in \mathcal{L}_+^{(2)}\mathrm{GL}(H)$; then it intertwines with $J_\nu^{-1}(t)$ by*

$$J_\nu^{-1}(t) \circ \widehat{E}(z) = \widehat{E}(j_\nu^2(t) z) \circ J_\nu^{-1}(t).$$

Proof. From (8.8) and the definition of $J_\nu^{-1}(t)\circ$, we know that the pairing η is scaled by $j_\nu^2(t)$ when applying $J_\nu^{-1}(t)\circ$. Thus the quadratic differential action $\widehat{E}(z)$ becomes $\widehat{E}(j_\nu^2(t) z)$. □

Anti-holomorphic completion and modular transformation.

Let $\mathcal{R}$ or $\mathcal{R}$ be a cohomology ring of any fixed Deligne-Mumford moduli space of stable curves of genus g with n marked points, i.e., $\mathcal{R} = H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ for some $2g - 2 + n > 0$.

Definition VIII.6. We say that a $\mathcal{R}$-valued function $f : \mathbb{H} \to \mathcal{R}$ is a $\mathcal{R}$-valued quasi-modular form of weight m with respect to some finite-index subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ if there are $\mathcal{R}$-valued functions f_i, $1 \leq i \leq K$, holomorphic on $\mathbb{H}$, such that

1. The functions $f_0 := f$ and f_i are holomorphic near cusp $\tau = i\infty$.

2. The following $\mathcal{R}$-valued function

$$f(\tau,\overline{\tau}) = f_0(\tau) + f_1(\tau)(\tau-\overline{\tau})^{-1} + \cdots + f_K(\tau)(\tau-\overline{\tau})^{-K}.$$

   is modular, i.e., there exists some $m \in \mathbb{N}$ such that for any $g \in \Gamma$,

$$f(g\tau, g\overline{\tau}) = j(g,\tau)^m f(\tau,\overline{\tau}).$$

$f(\tau,\overline{\tau})$ is called the *anti-holomorphic completion* of $f(\tau)$.

Anti-holomorphic completion of $\Lambda^W_{g,n}(t)$

Let W be the Fermat type cubic polynomial. Denote by

$$X_{t,\bar{t}}(z) = \begin{pmatrix} 1 & -z(t-\bar{t})^{-1} \\ 0 & 1 \end{pmatrix} \bigoplus I_6 \in \mathrm{End}(H)[[z]], \tag{8.11}$$

where $\bar{t}$ is the anti-holomorphic coordinate on $\mathbb{H}$ defined by

$$\bar{t} := \frac{a\bar{t}+b}{c\bar{t}+d}\,.$$

We define the *anti-holomorphic completion* of Coh FT $\Lambda^W(t)$ by:

$$\Lambda^W(t,\bar{t}) := \widehat{X}_{t,\bar{t}}(z) \circ \Lambda^W(t). \tag{8.12}$$

Theorem VIII.7. *Under the assumption of extension property, the analytic continuation of the anti-holomorphic completion $\Lambda^W_{g,n}(t,\bar{t})$ along ν is*

$$J_\nu^{-1}(t) \circ \Lambda^W_{g,n}(t,\bar{t}).$$

Proof. We define an operator $\widehat{X}_{\nu,t,\bar{t}}(z)$, s.t., the following diagram is commutative:

$$\begin{array}{ccc} \Lambda^W(t) & \xrightarrow{\widehat{X}_{t,\bar{t}}(z)} & \Lambda^W(t,\bar{t}) \\ \Big\downarrow{\scriptstyle J_\nu(t)\circ\widehat{X}_{\nu,t}(z)} & & \Big\downarrow{\scriptstyle \widehat{X}_{\nu,t,\bar{t}}(z)} \\ \Lambda^W(\nu(t)) & \xrightarrow{\widehat{X}_{\nu(t),\nu(\bar{t})}(z)} & \Lambda^W(\nu(t),\nu(\bar{t})) \end{array}$$

We need to prove that

$$\widehat{X}_{\nu,t,\bar{t}}(z) = J_\nu^{-1}(t).$$

Let us consider the analytic continuation for $X_{t,\bar{t}}(z)$. Analytic continuation acts on $(t-\bar{t})^{-1}$ by

$$\frac{1}{\nu(t)-\nu(\bar{t})} = -\Big(\frac{n_{12}}{j_\nu(t)} + \frac{1}{t-\bar{t}}\Big)\, j_\nu^2(t).$$

By definition (8.11), this implies

$$X_{\nu(t),\nu(\bar{t})}(z) = X_{t,\bar{t}}(j_\nu^2(t)z)\, X_{\nu,t}^{-1}(j_\nu^2(t)z). \tag{8.13}$$

Recalling Lemma VIII.5, we get,

$$J_\nu^{-1}(t)\circ \widehat{X}_{\nu,t}(z)\circ \widehat{X}_{t,\bar{t}}^{-1}(z) = \widehat{X}_{\nu,t}(j_\nu^2(t)z)\circ \widehat{X}_{t,\bar{t}}^{-1}(j_\nu^2(t)z)\circ J_\nu^{-1}(t). \tag{8.14}$$

Thus the result follows from (8.13) and (8.14). □

Cycle-valued quasi-modular forms from $\Lambda_{g,n}^W(t)$

We consider a pair

$$(\vec{\gamma}_I, \iota_I) = \big((\gamma_1,\cdots,\gamma_n),(\iota_1,\cdots,\iota_n)\big)\in H^{\otimes n}\times \mathbb{Z}_{\geq 0}^n$$

where each $\gamma_i\in\mathscr{S} = \{\partial_{-1}=\partial_{\mu-1},\partial_0,\cdots,\partial_{\mu-2}\}$. I is a multi-index

$$I = (i_{-1}, i_0,\cdots, i_{\mu-2})\in\mathbb{Z}_{\geq 0}^{\mu},\quad i_{-1}+\cdots+i_{\mu-2} = n.$$

i_j is the number of $i\in\{1,\cdots,n\}$ such that $\gamma_i=\partial_j$. Under the assumption of extension property, we define a cycle-valued function f_{I,ι_I}^W on $\mathbb{H}$,

$$f_{I,\iota_I}^W(t) = \Lambda_{g,n}^W(t)(\vec{\gamma}_I)\ \in H^*(\overline{\mathcal{M}}_{g,n},\mathbb{C}). \tag{8.15}$$

and its anti-holomorphic completion

$$f_{I,\iota_I}^W(t,\bar{t}) := \Lambda_{g,n}^W(t,\bar{t})(\vec{\gamma}_I).$$

For $\iota_I = (0, \cdots, 0)$, we simply denote them by $f_I^W(t)$ and $f_I^W(t, \bar{t})$. Let

$$m(I) := 2i_{-1} + \sum_{j=1}^{\mu-2} i_j. \tag{8.16}$$

Proposition VIII.8. *Let W be a simple elliptic singularity. Then $f_{I,\iota_I}^W(t)$ satisfies the transformation law of cycle-valued quasi-modular forms of weight $2g - 2 + m(I)$.*

Proof. First we consider $\iota_I = (0, \cdots, 0)$. It is easy to see $f_I^W(t, \bar{t})$ is an anti-holomorphic completion for $f_I^W(t)$ and for monodromy ν described as before, we have

$$\begin{aligned} f_I^W(\nu(t), \nu(\bar{t})) &= (\widehat{X}_{\nu,t,\bar{t}}(z) \circ \Lambda^W(t, \bar{t}))_{g,n}(\vec{\gamma}_I) \\ &= j_\nu^{2g-2+m(I)}(t)\, \Lambda_{g,n}^W(t, \bar{t})(\vec{\gamma}_I) \\ &= j_\nu^{2g-2+m(I)}(t)\, f_I^W(t, \bar{t}). \end{aligned}$$

The factor $2g - 2$ comes from the rescaling of $\hbar$. Now the statement follows from monodromy acts trivially on ψ-classes. □

Remark VIII.9. For $f_I^W(t)$ to be a cycle-valued modular form, it needs to be holomophic at $\tau = \sqrt{-1}\,\infty$. This will be achieved by the mirror theorem in the next section. Hence, by combining A-model with B-model, we produce cycle-valued quasi-modular forms.

8.4 Mirror symmetry

We identify via the mirror map the flat coordinates $\mathbf{t}^B$ on $\mathcal{M}$ and the linear coordinates $\mathbf{t}$ on D_ϵ. Recall the CohFT $\Lambda_{g,n}^W(\mathbf{t}^B)$ defined by formula (8.5) for all semisimple points $\mathbf{t}^B$.

Proposition VIII.10. *The CohFT $\Lambda_{g,n}^W(\mathbf{t}^B)$ extends holomorphically for all $\mathbf{t}^B \in D_\epsilon^B$, the ancestor Gromov–Witten CohFT ${}_{\mathbf{t}}\Lambda^X$ is convergent for all $\mathbf{t} \in D_\epsilon$, and we have*

$${}_{\mathbf{t}}\Lambda_{g,n}^X = \Lambda_{g,n}^W(\mathbf{t}^B), \quad \forall t \in D_\epsilon.$$

Proof. The Frobenius structure of the quantum cohomology is generically semi-simple. In particular, if we think of the CohFT ${}_{\mathbf{t}}\Lambda^{\mathcal{X}}$ as a CohFT over the field

$$\overline{\operatorname{Frac}\mathbb{C}[[e^t, t_0, \cdots, t_{\mu-2}]]},$$

where overline means *algebraic closure* and Frac stands for the *field of fractions*; then ${}_{\mathbf{t}}\Lambda^{\mathcal{X}}$ is a semi-simple CohFT with a flat identity. Teleman's reconstruction Theorem VIII.1 applies and we get that

$$(8.17)\qquad {}_{\mathbf{t}}\Lambda^{\mathcal{X}}_{g,n} = \Lambda^{W}_{g,n}(\mathbf{t}^B),$$

where the equality should be interpreted as equality in the space

$$H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C}) \otimes \overline{\operatorname{Frac}\mathbb{C}[[e^t, t_0, \cdots, t_{\mu-2}]]}.$$

On the other hand, according to Lemma (VIII.2), $\Lambda^{W}_{g,n}(\mathbf{t}^B)$ is meromorphic for $\mathbf{t} \in D^B_\epsilon$, thus

$$(8.18)\qquad \Lambda^{W}_{g,n}(\mathbf{t}^B) = {}_{\mathbf{t}}\Lambda^{\mathcal{X}}_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C}) \otimes \overline{\operatorname{Frac}\mathbb{C}\{e^t, t_0, \cdots, t_{\mu-2}\}},$$

where $\mathbb{C}\{x_1, \ldots, x_n\}$ is the ring of convergent power series at $x_1 = \cdots = x_n = 0$ (the overline means algebraic closure). On the other hand, by definition

$$(8.19)\qquad {}_{\mathbf{t}}\Lambda^{\mathcal{X}}_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C}) \otimes \mathbb{C}[[e^t, t_0, \cdots, t_{\mu-2}]].$$

Now we apply the following lemma of Coates–Iritani,

Lemma VIII.11 ([CI1], Lemma 6.6)**.** *The intersection*

$$\overline{\operatorname{Frac}\mathbb{C}\{x_1, \cdots, x_n\}} \cap \mathbb{C}[[x_1, \cdots, x_n]] \subset \overline{\operatorname{Frac}\mathbb{C}[[x_1, \cdots, x_n]]}$$

coincides with $\mathbb{C}\{x_1, \cdots, x_n\}$.

This completes the proof. □

Extension property

In this subsection, we use Lemma 3.2 from [MR] to derive the extension property.

Proposition VIII.12. *The coefficients of $\Lambda^W_{g,n}(\mathbf{t}^B)(\gamma_1, \ldots, \gamma_n)$ extend holomorphically through $\mathcal{K}$, i.e., they are holomorphic functions on $\mathcal{M}$.*

Proof. Let us define an action of $\mathbb{C}^*$ on $\mathcal{M} = \mathbb{H} \times \mathbb{C}^{\mu-1}$ according to the weights of the coordinates $\mathbf{t}^B$. Since $\Lambda^W(\mathbf{t}^B)$ is a homogeneous CohFT, the domain $\widetilde{\mathcal{K}}$ of all $\mathbf{t}^B$ where the theory does *not* extend analytically is $\mathbb{C}^*$-invariant. Since $\widetilde{\mathcal{K}}$ is the set of points $\mathbf{t}^B \in \mathcal{M}$, such that $\Lambda^W(\mathbf{t}^B)$ has a pole, $\widetilde{\mathcal{K}}$ must be an analytic subset. Let us assume that $\widetilde{\mathcal{K}}$ is non-empty. The Hartogues extension theorem implies that the codimension of $\widetilde{\mathcal{K}}$ is at most 1 and hence precisely one. On the other hand, according to Theorem VIII.10, the polydisk D_ϵ is disjoint from $\widetilde{\mathcal{K}}$. In particular, $\mathbb{H} \times \{0\}$ is not contained in $\widetilde{\mathcal{K}}$ and hence the two subvarieties interesect transversely. This combined with the $\mathbb{C}^*$ invariance of $\widetilde{\mathcal{K}}$ implies that the connected components of $\widetilde{\mathcal{K}}$ have the form $\{\tau_0\} \times \mathbb{C}^{\mu-1}$. This is a contradiction, because $\widetilde{\mathcal{K}} \subset \mathcal{K}$, while $\{\tau_0\} \times \mathbb{C}^{\mu-1} \not\subset \mathcal{K}$. $\square$

Quasi-modularity

Finally, let us complete the proof of our main theorem. According to Theorem VIII.10 the Gromov–Witten CohFT $\Lambda^X_{g,n}(q)$ is convergent and it coincides with $\Lambda^W_{g,n}(\tau)$, under the mirror map. The latter transforms as a quasi-modular form according to Theorem VIII.8, it is analytic for all $\tau \in \mathbb{H}$ due to Proposition VIII.12, and finally it extends holomorphically over the cusp $\tau = i\infty$ because $\Lambda^X_{g,n}(q)$ extends holomorphically over $q = 0$. This completes the proof of Theorem I.8.

CHAPTER IX

Convergence

9.1 Convergence of Gromov-Witten theory

Now let us define the *length* of a genus-0 Gromov-Witten correlator.

Definition IX.1. We say the correlator is of *length* 0 if it contains an insertion $\mathcal{P}$. We say it is of *length* m ($m \geq 1$) if after applying at most m times WDVV equation, it can be reconstructed from linear products of length 0 correlators, and genus-0 correlators with fewer marked points or lower degree.

Let $I^{GW}_{0,n+3,d}(m)$ be the maximum absolute value of all genus-0 $(n+3)$-points primary correlators with degree d and length m. Let $I^{GW}_{0,n+3,d}(-1)$ be the maximum absolute value of $I_1(n) + I_2(n) + I_3(n)$ in the WDVV equation (2.16). Thus

$$I^{GW}_{0,n+3,d}(m) \leq 3I^{GW}_{0,n+3,d}(m-1) + I^{GW}_{0,n+3,d}(-1), m \geq 1. \tag{9.1}$$

By carefully using the reconstruction, we have the following estimation,

Lemma IX.2. *We have*

$$I^{GW}_{0,3,0} \leq 1, I^{GW}_{0,4,0} \leq 1/4.$$

For $d > 0$, we have:

$$I^{GW}_{0,3,d} \leq d^{-2}C^{d-1}, I^{GW}_{0,4,d} \leq C_0\, d^{-1}C^{d-1}.$$

Here C_0 and C are sufficient large constants such that $C_0^2 \ll C$.

Proof. The key idea of the estimation is using the algorithm. Essentially, we only need to prove the estimation for these basic correlators. The technique from basic correlators to non-basic correlators is actually the same as for Type 1 basic correlators in the following. Now, for each type of basic correlators, we can check for each degree d. It is easy to see the lemma holds true for $d \leq 1$. The computations are tedious but elementary.

- For Type 1 correlators, we apply (2.16). Assume M is the maximum length for Type 1 genus-0 4-point correlators. Thus inequality (9.1) implies

$$\begin{aligned} I_{0,4,d} &\leq 3^M I_{0,4,d}(0) + \frac{3^M - 1}{2} I_2(1) \\ &\leq 3^M d^{-1} C^{d-1} + (3^M - 1) 36 \sum_{i=1}^{d-1} (d-i)^{-2} i^{-1} C^{d-2} \\ &\leq \ C_0 d^{-1} C^{d-1}. \end{aligned}$$

- For Type 2 correlators, we apply the inequality (2.21),

$$\left| \langle \gamma', \gamma', \gamma, \gamma \rangle_{0,4,d} \right| = \left| \frac{|\beta|}{d} (I_1 + I_2) \right| \leq \frac{|\beta|}{d} (8\mu \, I_{0,4,d} I_{0,4,0} + \sum_{i=1}^{d-1} I_{0,4,d-i} I_{0,4,i})$$

- For Type 3 correlators, the equation (2.22) implies for Type3 correlators.

$$\left| \langle \alpha, \beta, \gamma \rangle_{0,3,d} \right| \leq \left| \frac{|\gamma|}{(d-1)} 72 \sum_{i=1}^{d-1} I_{0,4,d-i} I_{0,3,i} \right| \leq \frac{N^2}{C} d^{-2} C^{d-1}$$

  A similar technique also works for Type 4 and Type 5 correlators.

- For Type 6 correlators, the equation (2.24) implies

$$\left| \langle \alpha^j, \alpha, \alpha \rangle_{0,3,d} \right| \leq \frac{72|\beta|}{d^2} \sum_{i=1}^{d-1} \frac{1}{d-i} C^{d-i-1} \frac{C_0}{i} C^{i-1} \leq \frac{864 C_0}{C\, d^2} C^{d-1} \ \leq \ d^{-2} C^{d-1}.$$

□

Now we can continue on with more insertions,

Lemma IX.3. *For $n+d \geq 4$, we have:*

$$I^{GW}_{0,n,d} \leq \begin{cases} d^{n-5} C^{n+d-4}, & d \geq 1. \\ \\ C^{n-4}, & d = 0. \end{cases}$$

Proof. We take induction on n. Lemma IX.2 implies the estimation holds for $n \leq 4$. We assume it holds for $n \leq k+2$, where $k \geq 2$, and then prove it for $n = k+3$. For $k \geq 10$, according to degree formula (2.13), the correlator $\langle \alpha_1, \cdots, \alpha_n \rangle_{g,n,d}$ must contain some $\mathcal{P}$ insertion. Then induction holds by applying the divisor axiom. Thus we only need to verify for $k \leq 9$. We recall the terms in the WDVV equation (2.16). For $k \geq 2, d = 0$,

$$I^{GW}_{0,k+3,0} \leq \sum_{i=1}^{k-1} C_0 \binom{k}{i} I^{GW}_{0,3+i,0} I^{GW}_{0,k+3-i,0} \leq C_0\, 2^{k-1}\, C^{k-2} \leq C^{k-1}. \tag{9.2}$$

For $d \geq 1$, we calculate $I^{GW}_{0,k+3,d}(-1)$ and $I^{GW}_{0,k+3,d}(0)$ first. The divisor equation implies

$$I^{GW}_{0,k+3,d}(0) \leq d\, I^{GW}_{0,k+2,d} \leq d^{k-2}\, C^{k+d-2}. \tag{9.3}$$

Next, we have

$$\begin{aligned} \left|I_1(k)\right| &\leq 72 \sum_{j=1}^{k-1} \binom{k}{j} I^{GW}_{0,j+3,d}\, I^{GW}_{0,k+3-j,0} \leq 72 \sum_{j=1}^{k-1} \binom{k}{j} d^{j-2} C^{k+d-2} \leq 72 \cdot 2^k d^{k-2} C^{k+d-2}, \\ \left|I_2(k)\right| &\leq 36 \sum_{i=1}^{d-1} \sum_{j=0}^{k} \binom{k}{j} I^{GW}_{0,j+3,d-i}\, I^{GW}_{0,k+3-j,i} \leq 36 \sum_{i=1}^{d-1} \sum_{j=0}^{k} \binom{k}{j} \big(d-i\big)^{j-2} i^{k-j-2} C^{k+d-2} \\ &\leq 288\, d^{k-2} C^{k+d-2}, \\ \left|I_3(k)\right| &\leq 72\, I^{GW}_{0,3,d}\, I^{GW}_{0,k+3,0} \leq 72 d^{-2} C^{k+d-2}. \end{aligned}$$

For the estimation of $I_2(k)$, we use for any $1 \leq i \leq d$,

$$\sum_{j=1}^{k-1} \binom{k}{j} \Big(\frac{i}{d}\Big)^j \Big(\frac{d-i}{d}\Big)^{k-j} \leq 1. \tag{9.4}$$

and

$$\sum_{i=1}^{d-1} i^{-2} (d-i)^{-2} \leq 6\, d^{-2}. \tag{9.5}$$

Now we have,

$$I^{GW}_{0,k+3,d}(-1) \leq \Big| I_1(k) + I_2(k) + I_3(k) \Big| \leq 18 \frac{2^{k+2} + 16 + 4}{C} d^{k-2} C^{k+d-1}.$$

Again, the length is bounded by some $M \leq 72$. Using (9.1) repeatedly, we know

$$I^{GW}_{0,k+3,d} \leq 3^M I^{GW}_{0,k+3,d}(0) + \frac{3^M - 1}{2} I^{GW}_{0,k+3,d}(-1) \leq d^{k-2} C^{k+d-1}.$$

□

Lemma IX.4. *For genus-1 primary correlators, for $n \geq 1$, we have*

$$I^{GW}_{1,n,d} \leq \begin{cases} d^{2n-3}\, C^{n+2d-2}, & \text{if } d > 0. \\ \\ 1, & \text{if } d = 0. \end{cases}$$

Proof. The non-vanishing terms are just $\langle \mathcal{P}, \cdots, \mathcal{P} \rangle^{\mathcal{X}}_{1,n,d}$. For $d = 0$, it is easy. For $d > 0$, the divisor axiom implies we only need to verify for $n = 1$. We prove the estimation for the $\mathcal{X} = \mathbb{P}^1_{3,3,3}$ case. The other two cases is similar. When we integrate $\Lambda^{\mathbb{P}^1_{3,3,3}}_{1,4,d+1}(\Delta_{2,2;1;1})$ over Getzler's relation, the genus-1 contribution only comes from $\delta_{3,4}$. Recall (2.15), for $d \geq 3$,

$$\begin{aligned} \Big| \sum_{i=0}^{d-1} \langle \mathcal{P} \rangle_{1,1,i} \langle \Delta_{2,2;1;1} \rangle_{0,4,d+1-i} \Big| &\leq \sum_{i=1}^{d-1} i^{-1} C^{2i-1} \Big(d + 1 - i\Big)^{-1} C^{d+1-i} + \Big(24d\Big)^{-1} C^{d+1} \\ &\leq \Big(6d\Big)^{-1} C^{2d-1}. \end{aligned}$$

We also have

$$\begin{aligned} \Big| \int_{\delta_\beta} \Lambda^{\mathbb{P}^1_{3,3,3}}_{1,4,d+1}(\Delta_{2,2;1;1}) \Big| &\leq \sum_{i=0}^{d+1} I^{GW}_{0,4,i} I^{GW}_{0,4,d+1-i} \leq \sum_{i=1}^{d+1} i^{-1} \Big(d + 1 - i\Big)^{-1} C^{2d} + 2\Big(d + 1\Big)^{-1} C^{d+1} \\ &\leq d^{-1} C^{2d+1}. \end{aligned}$$

Other genus-0 contributions are all bounded by $d^{-1} C^{2d+1}$. Thus the estimation follows from the Getzler relation. □

Now we give a proof of Theorem II.11.

Proof. Recall the expression of the ancestors,

$$\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n) \rangle_{g,n,d} = \int_{\overline{\mathcal{M}}_{g,n}} \Psi_{l_1,\cdots,l_n} \cdot \Lambda^{\mathcal{X}}_{g,n,d}(\alpha_1, \cdots, \alpha_n),$$

where $\Psi_{l_1,\cdots,l_n} = \prod_i \bar{\psi}_i^{l_i}$. We denote by $\deg \Psi_{l_1,\cdots,l_n} = \sum_i l_i := L$. Now we use g-reduction. According to degree counting for elliptic orbifold $\mathbb{P}^1$, the integration can be expressed as linear combinations of primary correlators with genus 0 or 1,

$$\int_{\overline{\mathcal{M}}_{g,n}} \Psi_{l_1,\cdots,l_n} \cdot \Lambda^{\mathcal{X}}_{g,n,d}(\alpha_1, \cdots, \alpha_n) = \sum_{j=1}^{N} \int_{\Gamma_j} \Lambda^{\mathcal{X}}_{g,n,d}(\alpha_1, \cdots, \alpha_n).$$

Here Γ_j is a connected dual graph with L edges. N is the number of such dual graphs. It depends only on genus g and number of marked points n. We denote the number of components in Γ_j by $L(j)$. Thus $1 \le L(j) \le L$. Each component of Γ_j is either of genus-0 or genus-1. Let k_i the number of nodes on the i-th component of Γ_j. Thus we know $k_i \ge 1, \sum_i k_i = 2L$. For each $\Gamma_j, \chi(j) = \chi$. Again, we have

$$\chi(j) = \sum_{i=1}^{L(j)} \chi_i.$$

We simply denote $C(\chi)$ by C if there is no confusion.

If $L = 0$, (2.26) follows from the previous three lemmas. Moreover, in $d = 0$ case, the absolute value is bounded by $C^{\chi-2}$.

For $L \ge 1$, $d = 0$,

$$\left| \int_{\Gamma_j} \Lambda^{\mathcal{X}}_{g,n,0}(\alpha_1, \cdots, \alpha_n) \right| \le 6^L \prod_{i=1}^{L(j)} I^{GW}_{g_i,n_i+k_i,0} \le 6^L \prod_{i=1}^{L(j)} C^{\chi_i - 2} \le 6^L C^{\chi - 2L(j)},$$

where 6^L is the upper bound for the products of pairing factors $\eta^{\mu,\nu}$ from L nodes. Thus

$$\left| \langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n) \rangle_{g,n,0} \right| \le \frac{6^L N}{C} C^{\chi-1} \le C^{\chi-1}.$$

For $d \ge 1$, we need to deal with those terms with $d_i = 0$, as the exponent of $C(\chi)$ in the estimation will increase by 1 in these cases. We generalize (9.5) and will have the following inequality:

$$\sum_{d_1+\cdots+d_{L(j)}=d} \prod_{d_i\neq 0} d_i^{\chi_i-2} \leq 8^L d^{\chi-2}.$$

Let K be the number of d_i which is 0, then $K \leq L(j)$. Now we have the estimation,

$$\begin{aligned}
\Big| \int_{\Gamma_j} \Lambda^{\mathcal{X}}_{g,n,d}(\alpha_1, \cdots, \alpha_n)\Big| &\leq \sum_{d_1+\cdots+d_{L(j)}=d} 6^L \prod_{i=1}^{L(j)} I^{GW}_{g_i,n_i+k_i,d_i} \\
&\leq \sum_{d_1+\cdots+d_{L(j)}=d} 6^L C^K \prod_{d_i\neq 0} d_i^{\chi_i-2} \prod_{i=1}^{L(j)} C^{\chi_i+(g_i+1)d_i-2} \\
&\leq 6^L 8^L d^{\chi-2} C^{\chi+(g+1)d-2}.
\end{aligned}$$

Now (2.26) follows from $L \cdot d \neq 0$ in this case. □

9.2 Convergence of FJRW theory

For the elliptic singularity (W, G_W), recall that the *FJRW ancestor correlator function*:

$$\langle\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n)\rangle\rangle^W_{g,n}(\mathbf{t}) = \sum_{k=0} \frac{1}{k!} \langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n), \mathbf{t}, \cdots, \mathbf{t}\rangle^W_{g,n+k},$$

where $\mathbf{t} \in H^{FJRW}_W$ (include the complex degree one case). In this subsection, we prove the convergence of the functions near $\mathbf{t} = 0$. We first define the *length* for a FJRW genus-0 primary correlator.

Definition IX.5. We say a genus-0 n-points FJRW correlator has *length m* if it can be reconstructed by genus-0 FJRW correlators with fewer marked points by at most $m + 1$ WDVV equations.

Recall the insertions α_i belong to the basis we fixed. We denote

$$\begin{aligned}
&I_{g,n} := \max\left\{\left|\langle \alpha_1, \cdots, \alpha_n\rangle^W_{g,n}\right|\right\}, \\
(9.6)\qquad &I_{0,n}(m) := \max\left\{\left|\langle \alpha_1, \cdots, \alpha_n\rangle^W_{0,n}\right| \Big| \langle \alpha_1, \cdots, \alpha_n\rangle^W_{0,n} \text{ is of length } m\right\}, \\
&I_{g,n,k,L} := \max\left\{\Big| \int_{\overline{\mathcal{M}}_{g,n+k}} \pi^*_{n,k}(\Psi_{g,n,L}) \cdot \Lambda^W_{g,n+k}(\alpha_1, \cdots, \alpha_n, \rho_{-1}, \cdots, \rho_{-1})\Big|\right\}.
\end{aligned}$$

Here $\Psi_{g,n,L}$ is a monomial of ψ and κ classes in $H^*(\overline{\mathcal{M}}_{g,n})$ with $\deg \Psi_{g,n,L} = L$. The Selection rule (3.5) implies nonzero integrals, L is bounded by g and n,

$$2g-2 \leq L \leq 2g-2+n.$$

On the other hand, for $m \geq 0$, the WDVV equation (3.13) implies

$$I_{0,n}(m) \leq 12 I_{0,n}(m-1) + 2|I(n-3)|, \tag{9.7}$$

where we have the convention $I_{0,n}(-1) = 0$.

Lemma IX.6. *For $K \leq 4$, $I_{0,K} \leq C_0$ for some C_0. For $K \geq 5$, there exists sufficient large constant C, such that*

$$I_{0,K} \leq C^{K-4}(K-5)! \tag{9.8}$$

Proof. For n fixed, Selection rule (3.5) implies $\langle \alpha_1, \cdots, \alpha_n \rangle^W_{0,n}$ has at most 12 insertions other than ρ_{-1}. On the other hand, it is easy to see a genus-0 correlator with at least three ρ_{-1} insertions has length 0. Every step of WDVV equation (3.13) will decrease the degree of one non-primitive insertion. Thus the length of $\langle \alpha_1, \cdots, \alpha_n \rangle^W_{0,n}$ is bounded by some constant M. Thus the formula (9.7) implies

$$I_{0,n} \leq I_{0,n}(M) < 12^{M+1}|I(n-3)|.$$

This shows we only need to estimate the values of those correlators with fewer insertions. We use induction on the number of insertions. For $K \leq 5$, the estimation holds as there are just finite different correlators. Assume the estimation (9.8) holds for all $K \leq k+2$, $k \geq 4$, then the induction is true by the following estimation,

$$\begin{aligned}
|I(k)| &\leq 12 \sum_{i=2}^{k-2} \binom{k}{i} I_{0,i+3} I_{0,k-i+3} + 24k\, I_{0,4} I_{0,k+2} \\
&\leq \Big(12 \sum_{i=2}^{k-2} \frac{k(k-1)}{i(i-1)(k-i)(k-i-1)} + \frac{24kC_0}{k-2}\Big) C^{k-2}(k-2)! \\
&\leq \frac{54+48C_0}{C} C^{k-1}(k-2)!
\end{aligned}$$

Here we use the inequality

$$\sum_{i=2}^{k-2}\frac{k(k-1)}{i(i-1)(k-i)(k-i-1)}=\frac{k}{k-1}\sum_{i=2}^{k-2}(\frac{1}{i-1}+\frac{1}{k-i})(\frac{1}{i}+\frac{1}{k-i-1})\leq\frac{9}{2}.$$

□

Lemma IX.7. *For genus-1 primary correlators, we have*

$$I_{1,K}\leq C^K K!$$

Proof. We only give the proof of P_8 case, i.e. for $I^{P_8}_{1,K}$. The other two cases are similar. It is easy to see the estimation holds for $K=1$. Thus we can use the method of induction, assume it holds for $K\leq k+1$, $k\geq 0$. In this section, we simplify the notation by $\Lambda:=\Lambda^{P_8}_{1,k+4}(\mathbf{e}_x,\mathbf{e}_{yz},\mathbf{e}_y,\mathbf{e}_{xz},\mathbf{e}_{xyz},\cdots,\mathbf{e}_{xyz})$. Recall

$$\langle\mathbf{e}_{xyz},\cdots,\mathbf{e}_{xyz}\rangle^{P_8}_{1,k+2}=\frac{1}{3}\int_{\pi^{-1}_{4,k}(\delta_{2,2})}\Lambda$$

The integration of Λ on $\pi^{-1}_{4,k}(\delta_{2,3})$ and $\pi^{-1}_{4,k}(\delta_{2,4})$ are both zero. And we also have

$$\Big|\int_{\pi^{-1}_{4,k}(\delta_{3,4})}\Lambda\Big|\leq 4\sum_{i=0}^{k}\binom{k}{i}I_{1,i+1}I_{0,3}I_{0,k-i+4}\leq\frac{4C_0+4C_0^2}{C}C^{k+2}(k+2)!\leq C^{k+2}(k+2)!$$

Next, we consider the genus-0 contribution. Similarly, we have

$$\Big|\int_{\pi^{-1}_{4,k}(\delta_{0,3}+\delta_{0,4}-2\delta_\beta)}\Lambda\Big|\leq C^{k+2}(k+2)!$$

Now we integrate Λ on $\pi^{-1}_{4,k}(12\delta_{2,2}+4\delta_{2,3}-2\delta_{2,4}+6\delta_{3,4}+\delta_{0,3}+\delta_{0,4}-2\delta_\beta)$. Combine all the inequalities above, Getzler's relation implies

$$I^{P_8}_{1,k+2}=\max\Big|\langle\mathbf{e}_{xyz},\cdots,\mathbf{e}_{xyz}\rangle^{P_8}_{1,k+2}\Big|\leq C^{k+2}(k+2)!$$

□

Lemma IX.8. *Using g-reduction, for $K=n+k$, $L=\sum_{i=1}^n l_i$, we have*

$$\Big|\langle\tau_{l_1}(\alpha_1),\cdots,\tau_{l_n}(\alpha_n),\rho_{-1},\cdots,\rho_{-1}\rangle^W_{g,K}\Big|\leq I_{g,n,k,L}\leq C(\chi)^{2g-2+K+L}(2g-2+K+L)!$$

Here $C(\chi)$ is a sufficiently large constant depends increasingly on $\chi=2g-2+n$.

Proof. For fixed g and n, we can use induction on L. In case of $L = 0$, non-vanishing correlators must be of genus-0 or genus-1, thus the estimation follows from two lemmas above. For $L \geq 1$, according to g-reduction, $\Psi_{l_1,\cdots,l_n} := \prod_{i=1}^n \psi_i^{l_i}$ can be represented by a linear combination of dual graphs, each of which has at least one edge. The number of dual graphs is only depends on g and n, but not k. We denote by $N_{g,n}$. We can choose $C(\chi)$ such that $C(\chi) > N_{g,n}$.

We can induct on the number of edges in the dual graph. For one edge case, if there are two components, C_i, $i = 1, 2$. component C_i has genus g_i, n_i number of insertions from the n insertion, and the degree of ψ, κ classes is L_i. Those are all fixed by the dual graph. However k copies of insertion ρ_{-1} can be distributed to either C_1 or C_2. We denote the number of copies in C_i by k_i. Thus we have

$$g = g_1 + g_2, n = n_1 + n_2, k = k_1 + k_2, L_1 + L_2 = L - 1, \chi = \chi_1 + \chi_2. \tag{9.9}$$

Now, $C(\chi_1), C(\chi_2) < C(\chi)$. As $L_i < L$, by induction, we have the total bound

$$\begin{aligned}
&N_{g,n} \sum_{k_1=0}^{k} \binom{k}{k_1} I_{g_1,n_1+1,k_1,L_1}\, I_{g_2,n_2+1,k_2,L_2} \\
&\leq N_{g,n} C(\chi)^{2g-3+n+k+L} \sum_{k_1=0}^{k} \binom{k}{k_1} (2g_1 - 1 + n_1 + k_1 + L_1)!\,(2g_2 - 1 + n_2 + k_2 + L_2)! \\
&= N_{g,n} C(\chi)^{2g-3+n+k+L} \frac{(2g_1 - 1 + n_1 + L_1)!(2g_2 - 1 + n_2 + L_2)!}{(2g - 2 + n + L)!} (2g - 2 + n + k + L)! \\
&\leq C(\chi)^{2g-2+n+k+L} (2g - 2 + n + k + L)!
\end{aligned}$$

The second equality is using the Chu-Vandemonde equality:

$${}_2F_1(-k, b; c; 1) = \frac{(c-b)_k}{(c)_k} = \frac{(c-b)(c-b+1)\cdots(c-b+k-1)}{(c)(c+1)\cdots(c+k-1)}, \tag{9.10}$$

where we set $b = 2g_1 + n_1 + L_1, c = -k - 2g_2 - n_2 - L_2 + 1$.

If there is just one component, χ is invariant, then the total bound is

$$N_{g,n} I_{g-1,n+2,k,L-1} \leq N_{g,n} C(\chi)^{\chi+k+L-1} (\chi + k + L - 1)! \leq C(\chi)^{2g-2+K+L} (2g - 2 + K + L)!$$

□

Now we give the proof of Lemma (III.13).

Proof. By definition and the previous lemma, we have

$$
\begin{aligned}
&\left|\langle\langle\tau_{l_1}(\alpha_1),\cdots,\tau_{l_n}(\alpha_n)\rangle\rangle_{g,n}^{W,G}(s\rho_{-1})\right| \\
&=\Big|\sum_{k=0}\frac{1}{k!}\langle\tau_{l_1}(\alpha_1),\cdots,\tau_{l_n}(\alpha_n),s\rho_{-1},\cdots,s\rho_{-1}\rangle_{g,n+k}^{W,G}\Big| \\
&\leq\sum_{k=0}^{\infty}C(\chi)^{2g-2+n+k+L}\frac{(2g-2+n+k+L)!}{k!}s^k.
\end{aligned}
$$

Thus the ancestor function is convergent in $|s|<\frac{1}{2C(\chi)}$. □

APPENDIX

APPENDIX A

Recursion formulas for basic corrlators in Gromov-Witten theory

A.1 A list of basic correlators for $\mathbb{P}^1_{3,3,3}$

Let us define:

$$A_1(k) = \langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{1,1} \rangle_{0,3,3k} \qquad A_2(k) = \langle \Delta_{1,2}, \Delta_{1,1}, \Delta_{2,2}, \Delta_{2,1} \rangle_{0,4,3k}$$

$$A_3(k) = \langle \Delta_{1,2}, \Delta_{1,2}, \Delta_{1,1}, \Delta_{1,1} \rangle_{0,4,3k} \qquad A_4(k) = \langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle_{0,3,3k+1}$$

$$A_5(k) = \langle \Delta_{1,2}, \Delta_{1,2}, \Delta_{2,1}, \Delta_{3,1} \rangle_{0,4,3k+1} \qquad A_6(k) = \langle \Delta_{1,2}, \Delta_{2,2}, \Delta_{3,1}, \Delta_{3,1} \rangle_{0,4,3k+2}$$

Then we obtain the following recursion formulas from WDVV equations:

$$\begin{aligned}
A_2(k) &= 3\sum_{i=0}^{k-1} A_4(k-i-1)A_6(i) - kA_1(k) - 3\sum_{i=1}^{k-1} A_1(k-i)A_2(i) \\
kA_3(k) &= -6\sum_{i=1}^{k} A_3(k-i)A_2(i) + 6\sum_{i=0}^{k} A_2(k-i)A_2(i) \\
kA_4(k) &= -3\sum_{i=1}^{k} A_4(k-i)A_3(i) + 3\sum_{i=1}^{k} A_4(k-i)A_2(i) \\
A_5(k) &= -3\sum_{i=1}^{k} A_1(i)A_5(k-i) + (k+1/3)A_4(k) + 3\sum_{i=1}^{k} A_4(k-i)A_2(i) \\
A_6(k) &= -3\sum_{i=0}^{k-1} A_1(k-i)A_6(i) + 3\sum_{i=0}^{k} A_4(k-i)A_5(i) \\
k^3A_1(k) &= -\sum_{i=1}^{k-1}(3k-3i)^2 A_1(k-i)A_2(i) + \sum_{i=0}^{k-1}(3k-3i-2)(3i+2)A_4(k-i-1)A_6(i)
\end{aligned}$$

A.2 A list of basic correlators for $\mathbb{P}^1_{4,4,2}$

In this case, we define

$$A_1(k) = \langle \Delta_{1,2}, \Delta_{1,1}, \Delta_{1,1} \rangle_{0,3,4k}; \qquad A_2(k) = \langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{1,3}, \Delta_{1,1} \rangle_{0,4,4k};$$

$$A_3(k) = \langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{1,2}, \Delta_{1,2} \rangle_{0,4,4k}; \qquad A_4(k) = \langle \Delta_{1,3}, \Delta_{1,1}, \Delta_{2,3}, \Delta_{2,1} \rangle_{0,4,4k};$$

$$A_5(k) = \langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{3,1}, \Delta_{3,1} \rangle_{0,4,4k}; \qquad A_6(k) = \langle \Delta_{1,3}, \Delta_{1,3}, \Delta_{1,1}, \Delta_{1,1} \rangle_{0,4,4k};$$

$$A_7(k) = \langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle_{0,3,4k+1}; \qquad A_8(k) = \langle \Delta_{3,1}, \Delta_{1,3}, \Delta_{1,2}, \Delta_{2,1} \rangle_{0,4,4k+1};$$

$$A_9(k) = \langle \Delta_{1,2}, \Delta_{2,1}, \Delta_{2,1} \rangle_{0,3,4k+2}; \qquad A_{10}(k) = \langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{1,2}, \Delta_{2,2} \rangle_{0,4,4k+2};$$

$$A_{11}(k) = \langle \Delta_{1,3}, \Delta_{1,3}, \Delta_{2,1}, \Delta_{2,1} \rangle_{0,4,4k+2}; \quad A_{12}(k) = \langle \Delta_{3,1}, \Delta_{1,3}, \Delta_{2,2}, \Delta_{2,1} \rangle_{0,4,4k+3}.$$

Then the recursion formulas are:

$$\begin{aligned}
A_2(k) &= 4\sum_{i=1}^{k}(A_7(i-1)A_{12}(k-i) - A_1(i)A_2(k-i)) - 2kA(k) \\
A_3(k) &= 4\sum_{i=1}^{k}(2A_7(i-1)A_{12}(k-i) - A_9(i-1)A_{10}(k-i) - A_1(i)A_3(k-i)) - 2kA_1(k) \\
A_4(k) &= 4\sum_{i=1}^{k}(-A_1(i)A_4(k-i)) - kA_1(k) \\
kA_5(k) &= 4\sum_{i=1}^{k}(-A_3(i)A_5(k-i) + 2A_3(i)A_3(k-i) + 2A_{10}(i-1)A_{10}(k-i)) - 2kA_3(k) \\
2kA_6(k) &= 4\sum_{i=1}^{k}(-2A_2(i)A_6(k-i) + A_2(i)A_2(k-i)) \\
\frac{4k+1}{2}A_7(k) &= -2A_7(k)A_5(0) + 2\sum_{i=1}^{k}(4A_2(i) - A_5(i))A_7(k-i) \\
A_8(k) &= 4\sum_{i=1}^{k}(-A_1(i)A_8(k-i) - A_9(i-1)A_{12}(k-i) + 2A_4(i)A_7(k-i)) \\
(2k+1)A_9(k) &= -4\sum_{i=1}^{k}A_2(i)A_9(k-i) + 4\sum_{i=0}^{k}A_7(i)A_8(k-i)
\end{aligned}$$

$$
\begin{aligned}
A_{10}(k) &= -4\sum_{i=1}^{k} A_1(i)A_{10}(k-i) + 4\sum_{i=0}^{k}(A_7(i)A_8(k-i) + A_2(i)A_9(k-i) - A_3(i)A_9(k-i)) \\
A_{11}(k) &= -4\sum_{i=1}^{k} A-1(i)A_{11}(k-i) + 2\sum_{i=0}^{k} A_7(i)A_8(k-i) \\
A_{12}(k) &= -4\sum_{i=1}^{k} A_1(i)A_{12}(k-i) + 4\sum_{i=0}^{k}(A_7(i)A_{11}(k-i) - A_8(i)A_9(k-i)) \\
k^2A_1(k) &= -4\sum_{i=1}^{k-1} iA_1(i)A_2(k-i) + \sum_{i=1}^{k}(4i-3)A_7(i-1)A_{12}(k-i)
\end{aligned}
$$

A.3 A list of basic correlators for $\mathbb{P}^1_{6,3,2}$

In this case, we define

$A_1(k) = \langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{1,4} \rangle_{0,3,6k}$; $A_2(k) = \langle \Delta_{1,1}, \Delta_{1,2}, \Delta_{1,3} \rangle_{0,3,6k}$;

$A_3(k) = \langle \Delta_{2,1}, \Delta_{2,1}, \Delta_{2,1} \rangle_{0,3,6k}$; $A_4(k) = \langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{1,1}, \Delta_{1,5} \rangle_{0,4,6k}$;

$A_5(k) = \langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{1,2}, \Delta_{1,4} \rangle_{0,4,6k}$; $A_6(k) = \langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{1,3}, \Delta_{1,3} \rangle_{0,4,6k}$;

$A_7(k) = \langle \Delta_{2,1}, \Delta_{1,1}, \Delta_{2,2}, \Delta_{1,2} \rangle_{0,4,6k}$; $A_8(k) = \langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{2,1}, \Delta_{2,2} \rangle_{0,4,6k}$;

$A_9(k) = \langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{1,5}, \Delta_{1,5} \rangle_{0,4,6k}$; $A_{11}(k) = \langle \Delta_{2,1}, \Delta_{2,1}, \Delta_{2,2}, \Delta_{2,2} \rangle_{0,4,6k}$;

$A_{12}(k) = \langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{3,1}, \Delta_{3,1} \rangle_{0,4,6k}$; $A_{10}(k) = \langle \Delta_{3,1}, \Delta_{2,1}, \Delta_{1,1} \rangle_{0,3,6k+1}$;

$A_{13}(k) = \langle \Delta_{3,1}, \Delta_{1,1}, \Delta_{2,2}, \Delta_{2,2} \rangle_{0,4,6k+1}$; $A_{14}(k) = \langle \Delta_{3,1}, \Delta_{2,1}, \Delta_{1,2}, \Delta_{1,5} \rangle_{0,4,6k+1}$;

$A_{15}(k) = \langle \Delta_{3,1}, \Delta_{2,1}, \Delta_{1,3}, \Delta_{1,4} \rangle_{0,4,6k+1}$; $A_{16}(k) = \langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{2,2} \rangle_{0,3,6k+2}$;

$A_{17}(k) = \langle \Delta_{2,1}, \Delta_{2,1}, \Delta_{1,2} \rangle_{0,3,6k+2}$; $A_{18}(k) = \langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{2,2}, \Delta_{1,2} \rangle_{0,4,6k+2}$;

$A_{19}(k) = \langle \Delta_{2,1}, \Delta_{2,1}, \Delta_{1,3}, \Delta_{1,5} \rangle_{0,4,6k+2}$; $A_{20}(k) = \langle \Delta_{2,1}, \Delta_{2,1}, \Delta_{1,4}, \Delta_{1,4} \rangle_{0,4,6k+2}$;

$$A_{21}(k) = \langle \Delta_{3,1}, \Delta_{1,1}, \Delta_{1,2} \rangle_{0,4,6k+3}; \qquad A_{22}(k) = \langle \Delta_{3,1}, \Delta_{1,1}, \Delta_{1,3}, \Delta_{1,5} \rangle_{0,4,6k+3};$$

$$A_{23}(k) = \langle \Delta_{3,1}, \Delta_{1,1}, \Delta_{1,4}, \Delta_{1,4} \rangle_{0,4,6k+3}; \quad A_{24}(k) = \langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{3,1}, \Delta_{1,3} \rangle_{0,4,6k+3};$$

$$A_{25}(k) = \langle \Delta_{3,1}, \Delta_{2,1}, \Delta_{2,2}, \Delta_{1,3} \rangle_{0,4,6k+3}; \quad A_{26}(k) = \langle \Delta_{2,1}, \Delta_{1,1}, \Delta_{1,3} \rangle_{0,4,6k+4};$$

$$A_{27}(k) = \langle \Delta_{2,1}, \Delta_{1,2}, \Delta_{1,2} \rangle_{0,4,6k+4}; \qquad A_{28}(k) = \langle \Delta_{3,1}, \Delta_{3,1}, \Delta_{2,1}, \Delta_{1,4} \rangle_{0,4,6k+4};$$

$$A_{29}(k) = \langle \Delta_{2,1}, \Delta_{2,1}, \Delta_{2,2}, \Delta_{1,4} \rangle_{0,4,6k+4}; \quad A_{30}(k) = \langle \Delta_{2,1}, \Delta_{1,1}, \Delta_{1,4}, \Delta_{1,5} \rangle_{0,4,6k+4};$$

$$A_{31}(k) = \langle \Delta_{3,1}, \Delta_{1,1}, \Delta_{2,2}, \Delta_{1,4} \rangle_{0,4,6k+5}; \quad A_{32}(k) = \langle \Delta_{3,1}, \Delta_{2,1}, \Delta_{2,1}, \Delta_{1,5} \rangle_{0,4,6k+5};$$

Then the recursion formulas are

$$\begin{aligned}
k^2 A_1(k) &= \sum_{i=0}^{k-1} \Big(\frac{6k-6i-5}{3} A_{31}(i) A_{10}(k-1-i) + (4k-4i-2) A_{23}(i) A_{21}(k-1-i) \Big) \\
&\quad - \sum_{i=1}^{k-1} (4k-4i) A_4(i) A_1(k-i) \\
k A_2(k) &= \sum_{i=0}^{k-1} \Big((3k-6i-4) A_{26}(i) A_{16}(k-1-i) + (6k-12i) A_2(i) A_1(k-i) \Big) \\
3k^2 A_3(k) &= \sum_{i=0}^{k-1} \Big((12k-6i-5) A_{32}(i) A_{10}(k-1-i) \\
&\quad -(12k-12i-8) A_{28}(i) A_{17}(k-1-i) \Big) - 6 \sum_{i=1}^{k-1} (k-i) A_8(i) A_3(k-i) \\
A_4(k) &= \sum_{i=0}^{k-1} \Big(3A_{31}(i) A_{10}(k-1-i) - 6A_4(i) A_1(k-i) + 6A_{23}(i) A_{21}(k-1-i) \Big) \\
&\quad -3k A_1(k) \\
A_5(k) &= \sum_{i=0}^{k-1} \Big(6A_4(i) A_1(k-i) - 6A_5(i) A_1(k-i) - 3A_{28}(i) A_{16}(k-1-i) \\
&\quad +3A_{31}(i) A_{10}(k-1-i) + 6A_{23}(i) A_{21}(k-1-i) \Big) + A_4(k) \\
A_6(k) &= \sum_{i=0}^{k-1} \Big(6A_5(i) A_2(k-i) - 6A_6(i) A_2(k-i) - 2A_{24}(i) A_{21}(k-1-i) \\
&\quad +3A_{26}(i) A_{18}(k-1-i) + 6A_{22}(i) A_{21}(k-1-i) \Big) + A_5(k)
\end{aligned}$$

$$
\begin{aligned}
A_7(k) &= \sum_{i=0}^{k-1}\Big(6A_{30}(i)A_{16}(k-1-i) - 6A_7(i)A_1(k-i)\Big) - 2kA_1(k) \\
A_8(k) &= \sum_{i=0}^{k-1}\Big(12A_{32}(i)A_{10}(k-1-i) - 3A_8(i)A_3(k-i) \\
&\quad -6A_{28}(i)A_{17}(k-1-i)\Big) - 3kA_3(k) \\
kA_9(k) &= \sum_{i=0}^{k-1}\Big(\frac{4}{3}A_4(i)A_4(k-i) - 4A_9(i)A_4(k-i) + 4A_{22}(i)A_{22}(k-1-i)\Big) \\
kA_{10}(k) &= \sum_{i=0}^{k-1}\Big(3A_{10}(i)A_7(k-i) - 6A_{10}(i)A_9(k-i) + 6A_{30}(i)A_{21}(k-1-i)\Big) \\
A_{11}(k) &= \sum_{i=0}^{k-1}\Big(\frac{2}{3}A_{10}(i)A_8(k-i) - A_{11}(i)A_{10}(k-i) + 2A_{26}(i)A_{25}(k-1-i) \\
&\quad -2A_{29}(i)A_{21}(k-1-i)\Big) - \frac{6k+1}{9}A_{10}(k) \\
A_{12}(k) &= \sum_{i=0}^{k-1}\Big(\frac{3}{2}A_{10}(i)A_8(k-i) - A_{12}(i)A_{10}(k-i) + 3A_{10}(i)A_4(k-i) \\
&\quad +3A_{28}(i)A_{21}(k-1-i) - 3A_{26}(i)A_{24}(k-1-i)\Big) - \frac{6k+1}{4}A_{10}(k) \\
A_{13}(k) &= \sum_{i=0}^{k-1}\Big(6A_{10}(i)A_7(k-i) - 3A_{13}(i)A_3(k-i) - 6A_{31}(i)A_{17}(k-1-i)\Big) \\
&\quad +\frac{6k+1}{3}A_{10}(k) \\
A_{14}(k) &= \sum_{i=0}^{k-1}\Big(3A_{10}(i)A_7(k-i) - 6A_{14}(i)A_1(k-i) - 3A_{32}(i)A_{16}(k-1-i) \\
&\quad +6A_{30}(i)A_{21}(k-1-i)\Big) + \frac{6k+1}{6}A_{10}(k) \\
A_{15}(k) &= \sum_{i=0}^{k-1}\Big(6A_{14}(i)A_1(k-i) - 6A_{15}(i)A_2(k-i) - 2A_{28}(i)A_{21}(k-1-i) \\
&\quad +6A_{30}(i)A_{21}(k-1-i)\Big) + A_{14}(k) \\
(3k+1)A_{16}(k) &= \sum_{i=0}^{k-1}\Big(3A_{13}(i)A_{10}(k-i) - 6A_{16}(i)A_4(k-i) + 6A_{31}(i)A_{21}(k-1-i)\Big) \\
&\quad +3A_{13}(k) \\
(3k+1)A_{17}(k) &= \sum_{i=0}^{k-1}\Big(6A_{14}(i)A_{10}(k-i) - 3A_{17}(i)A_8(k-i) - 6A_{28}(i)A_{27}(k-1-i) \\
&\quad +6A_{32}(i)A_{21}(k-1-i)\Big) + 6A_{14}(k)
\end{aligned}
$$

$$\begin{aligned}
A_{18}(k) &= \sum_{i=0}^{k-1}\Big(3A_{13}(i)A_{10}(k-i) - 6A_{18}(i)A_1(k-i) - 3A_{16}(i)A_8(k-i) \\
&\quad +6A_{16}(i)A_4(k-i) + 6A_{31}(i)A_{21}(k-1-i)\Big) + 3A_{13}(k) \\
A_{19}(k) &= \sum_{i=0}^{k-1}\Big(3A_{17}(i)A_7(k-i) - 6A_{19}(i)A_2(k-i) + 6A_{30}(i)A_{27}(k-1-i) \\
&\quad -2A_{32}(i)A_{21}(k-1-i)\Big) + \frac{3k+1}{3}A_{17}(k) \\
A_{20}(k) &= \sum_{i=0}^{k-1}\Big(6A_{19}(i)A_1(k-i) - 6A_{20}(i)A_2(k-i) + 6A_{30}(i)A_{26}(k-1-i) \\
&\quad -3A_{29}(i)A_{26}(k-1-i)\Big) + A_{19}(k) \\
(2k+1)A_{21}(k) &= \sum_{i=0}^{k-1}\Big(6A_{31}(i)A_{27}(k-1-i) - 6A_{21}(i)A_7(k-i)\Big) \\
&\quad + \sum_{i=0}^{k} 3A_{17}(i)A_{13}(k-i) \\
A_{22}(k) &= \sum_{i=0}^{k-1}\Big(-6A_{22}(i)A_2(k-i) - 2A_{21}(i)A_4(k-i)\Big) \\
&\quad + \sum_{i=0}^{k} 6A_{21}(i)A_9(k-i) + \frac{2k+1}{2}A_{21}(k) \\
A_{23}(k) &= \sum_{i=0}^{k-1}\Big(6A_{22}(i)A_1(k-i) - 6A_{23}(i)A_2(k-i) - 3A_{31}(i)A_{26}(k-1-i)\Big) \\
&\quad +A_{22}(k) \\
A_{24}(k) &= \sum_{i=0}^{k-1}\Big(6A_{21}(i)A_5(k-i) - 6A_{24}(i)A_2(k-i) + 6A_{21}(i)A_4(k-i)\Big) \\
&\quad + \sum_{i=0}^{k}\Big(3A_{18}(i)A_{10}(k-i) - 2A_{21}(i)A_{12}(k-i)\Big) \\
&\quad -\frac{6k+3}{2}A_{21}(k) \\
A_{25}(k) &= \sum_{i=0}^{k-1}\Big(-3A_{25}(i)A_3(k-i) + 6A_{32}(i)A_{26}(k-1-i)\Big) \\
&\quad + \sum_{i=0}^{k}\Big(6A_{19}(i)A_{10}(k-i) - 6A_{17}(i)A_{15}(k-i)\Big) \\
(3k+2)A_{26}(k) &= \sum_{i=0}^{k-1}\Big(6A_{22}(i)A_{10}(k-i) - 6A_{26}(i)A_4(k-i)\Big) + 6A_{22}(k)
\end{aligned}$$

$$
\begin{aligned}
A_{27}(k) &= \sum_{i=0}^{k}\Big(2A_{21}(i)A_{10}(k-i) - 3A_{17}(i)A_{16}(k-i) + 6A_{26}(i)A_{2}(k-i)\Big) \\
&\quad - \sum_{i=0}^{k-1} 6A_{27}(i)A_{1}(k-i) \\
A_{28}(k) &= \sum_{i=0}^{k}\Big(3A_{25}(i)A_{10}(k-i) - 3A_{26}(i)A_{8}(k-i) + 6A_{21}(i)A_{15}(k-i)\Big) \\
&\quad - \sum_{i=0}^{k-1} 6A_{28}(i)A_{2}(k-i) - (3k+2)A_{26}(k) \\
A_{29}(k) &= \sum_{i=0}^{k}\Big(6A_{26}(i)A_{7}(k-i) - 3A_{26}(i)A_{11}(k-i) + 6A_{19}(i)A_{16}(k-i)\Big) \\
&\quad - \sum_{i=0}^{k-1} 6A_{29}(i)A_{2}(k-i) - \frac{6k+4}{3}A_{26}(k) \\
A_{30}(k) &= \sum_{i=0}^{k}\Big(6A_{26}(i)A_{9}(k-i) - 3A_{26}(i)A_{7}(k-i)\Big) \\
&\quad - \sum_{i=0}^{k-1} 6A_{30}(i)A_{2}(k-i) + \frac{3k+2}{3}A_{26}(k) \\
A_{31}(k) &= \sum_{i=0}^{k}\Big(6A_{22}(i)A_{16}(k-i) - 3A_{26}(i)A_{13}(k-i)\Big) - \sum_{i=0}^{k-1} 6A_{31}(i)A_{2}(k-i) \\
A_{32}(k) &= \sum_{i=0}^{k}\Big(2A_{28}(i)A_{10}(k-i) + 6A_{26}(i)A_{15}(k-i) - 6A_{30}(i)A_{10}(k-i)\Big) \\
&\quad - \sum_{i=0}^{k-1} 6A_{32}(i)A_{1}(k-i)
\end{aligned}
$$

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