GENERALIZED HEEGNER CYCLES, SHIMURA CURVES, AND SPECIAL VALUES OF p-ADIC L-FUNCTIONS

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2013

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ACKNOWLEDGEMENTS

In addition to being a wise and extremely patient advisor, Kartik Prasanna suggested this problem to me and came up with a beautiful idea to treat the case where $\mathrm{wt}(f) = 2$ and χ gives the trivial character of $\mathrm{Gal}(H/K)$, which I had been excluding in a previous draft. It is impossible to sufficiently thank him. Ruochuan Liu read a draft of this document and offered several helpful comments. The portion of this document regarding the construction of Besser's projector benefitted from conversations with Kevin Carde and David Speyer. Finally, learning the necessary material required many conversations with fellow graduate students who did not need to learn it themselves, among whom I especially thank Ari Shnidman and Julian Rosen.

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CHAPTER I

Introduction

The theory of special values of L-functions links *geometric* invariants of algebraic cycles with values of functions that are defined purely *analytically.* An important tool for establishing results toward the many conjectural formulas in this area is the theory of Heegner points, as exemplified by the seminal work [GZ] of Gross and Zagier.

1.1 The Gross-Zagier formula and its generalizations

1.1.1 Two consequences of the Heegner hypothesis

Fix a normalized weight two newform f on $\Gamma_0(N)$ with trivial nebentypus and rational Hecke eigenvalues. Gross and Zagier consider an imaginary quadratic field K satisfying the so-called

> *Heegner hypothesis:* All primes q dividing N are split or ramified in K; moreover, if $q|N$ is ramified in K, then $q^2 \nmid N$.

This seemingly technical hypothesis in fact contributes in an essential way to both the geometric and analytic side of Gross and Zagier's famous result.

On the *geometric* side, the Heegner hypothesis, which is equivalent to the existence of an ideal $\mathcal{N}$ of K whose norm is N, implies the existence of a cyclic degree N-isogeny of complex elliptic curves

$$\frac{\mathbb{C}}{\mathcal{O}_K} \longrightarrow \frac{\mathbb{C}}{\mathcal{N}^{-1}},$$

that is, a point on the modular curve $X_0(N)(\mathbb{C})$. The classical theory of complex multiplication implies that this point is actually an H-point of $X_0(N)$, where H denotes the Hilbert class field of K. Writing $P \in X_0(N)(H)$ for this point (a "Heegner point"), we may consider its trace

$$P_{\mathbf{1}} := \sum_{\sigma \in \mathrm{Gal}(H/K)} P^{\sigma},$$

or, more generally, for a character $\chi : \mathrm{Gal}(H/K) \to \mathbb{C}^{\times}$, the twisted trace

$$P_{\chi} := \sum_{\sigma \in \mathrm{Gal}(H/K)} \chi^{-1}(\sigma) P^{\sigma}$$

(the symbol $\mathbf{1}$ denotes the trivial character).

Then P_{χ} is a divisor (with $\mathbb{C}$-coefficients) on $X_0(N)$, defined over H, in the χ-eigenspace for the action of $\mathrm{Gal}(H/K)$ of $\mathrm{Div}_H(X_0(N))$. There is an elliptic curve E and a modular parametrization

$$\phi : X_0(N) \to E$$

determined by f, normalized so as to map the cusp ∞ to the origin. Then $Q_{\chi} = \phi(P_{\chi})$ is a ($\mathbb{C}$-coefficient) divisor Q_{χ} on E, and in particular $Q_{\mathbf{1}} \in E(K)$ is a rational point.

On the *analytic* side, the Heegner hypothesis affects the vanishing behavior of the Rankin-Selberg L-function $L(f, \chi, s)$, where f denotes the newform attached to E. More precisely, the completed L-function $\Lambda(f, \chi, s)$ has a functional equation of the form

$$\Lambda(f, \chi, s) = \epsilon(f, \chi, s)\Lambda(f, \chi, 2 - s).$$

At the central point $s = 1$, the factor $\epsilon(f, \chi, 1) = \pm 1$ is a product of local signs $\epsilon_v(f, \chi)$, where v ranges over the places of $\mathbb{Q}$. At the "good reduction" places $v \nmid N$ the sign $\epsilon_v(f, \chi)$ is 1, while the sign at $v = \infty$ is

$$\epsilon_\infty(f, \chi) = -1,$$

which one can calculate as a consequence of a comparison of the weight of f (which is 2) and the weight of χ (which is 0). The Heegner hypothesis forces the local sign $\epsilon_v(f, \chi)$ to be 1 whenever $v|N$, and so the global sign $\epsilon(f, \chi, 1)$ is -1. Thus the Heegner hypothesis *forces* the vanishing of the L-function at $s = 1$.

Recall that $E(K)$ is equipped with a symmetric real-valued *height pairing* such that $\langle R, R \rangle = 0$ if and only if R is a torsion point of $E(K)$. Below, and in the rest of the introduction, the sign $\doteq$ indicates that two values are equal up to multiplication by a nonzero constant that we have omitted for simplicity (in each case, the explicit constant can be found in the cited reference). The following formula is Theorem I.6.1 of [GZ].

Theorem I.1 (Gross-Zagier)**.** *One has*

$$L'(f, \chi, 1) \doteq \langle Q_\chi, Q_\chi \rangle.$$

In the case $\chi = \mathbf{1}$, the Rankin-Selberg L-function coincides with the L-function $L(E/K, 1)$ of the base change of E to K. If the derivative of this L-function is nonzero (which implies that the analytic rank of E/K is exactly one), then the Gross-Zagier formula shows that the rank of $E(K)$ is at least one, confirming an implication of the Birch and Swinnerton-Dyer conjecture for K. As is explained on p. 312 of [GZ], this result descends to $\mathbb{Q}$ after incorporating a non-vanishing result of Waldspurger. Waldspurger's result ([Wa]) shows that one can pick K such that the pair (K, N) satisfies the Heegner hypothesis and the quadratic twist E' of E by the field K satisfies $L'(E'/\mathbb{Q}, 1) \neq 0$. It then follows from the factorization

$$L(E/K, s) = L(E/\mathbb{Q}, s)L(E'/\mathbb{Q}, s)$$

and an analysis of the constants in Theorem I.1 that

$$L'(E/\mathbb{Q}, 1) \doteq \langle \mathrm{Tr}_{K/\mathbb{Q}} Q_\chi, \mathrm{Tr}_{K/\mathbb{Q}} Q_\chi \rangle.$$

1.1.2 The p-adic Gross-Zagier formula

A p-adic variation on this theme was discovered by Perrin-Riou quickly following the announcement of [GZ]. In order to contextualize her work a little better, we consider an older result of Leopoldt. Consider the Dirichlet L-functions $L(s, \chi)$ as χ varies over Dirichlet characters of $\mathbb{Z}$. The values of $L(s, \chi)$ at negative integers are *algebraic*. Pick an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$ and compatible embeddings $\overline{\mathbb{Q}} \subset \mathbb{C}$ and $\overline{\mathbb{Q}} \subset \mathbb{C}_p$, and write ω for the $\mathbb{C}$-valued "Teichmuller character" of conductor p, given by

$$\mathbb{Z} \backslash p\mathbb{Z} \hookrightarrow \mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p) \longrightarrow \mu_{p-1} \subset \mathbb{C}_p^\times \simeq \mathbb{C}^\times.$$

In [KL], Kubota and Leopoldt established the existence of an analytic p-adic L-function $L_p^{\text{Dirichlet}}(\cdot, \chi)$, a function from $\mathbb{C}_p$ to $\mathbb{C}_p$ satisfying the interpolation property (for $n \geq 0$)

$$L_p^{\text{Dirichlet}}(1 - n, \chi) = (1 - \chi\omega^{-n}(p)p^{n-1})L(1 - n, \chi\omega^{-n}).$$

(Such a function is necessarily unique.) When χ is an odd character, the p-adic L-function identically vanishes. On the other hand, when χ is an even character, Leopoldt established the remarkable formula[1]

$$L_p(1, \chi) = -\left(1 - \frac{\chi(p)}{p}\right) \frac{\tau(\chi)}{m} \sum_{a=1}^{m} \chi^{-1}(a) \log_p(1 - \zeta_m^a).$$

[1]Leopoldt did not publish this formula; the first published proof is on p. 41 of [Iw].

Here, $\tau(\chi) \in \overline{\mathbb{Q}}$ denotes the usual Gauss sum and m denotes the conductor of χ; the map $\log_p$ is Iwasawa's p-adic logarithm and ζ_m is the image under our chosen isomorphism of $\exp(\frac{2\pi i}{m})$.

Leopoldt's formula is striking because it parallels the classical formula

$$L(1,\chi) = -\frac{\tau(\chi)}{m}\sum_{a=1}^{m}\chi^{-1}(a)\log(1-\zeta_m^a).$$

This result establishes a fundamental principle: any formula involving special values of a classical (motivic) L-function should have a p-adic analogue.

Returning to Perrin-Riou's p-adic variation on Gross-Zagier, suppose that p splits in K and is prime to N, and write K_∞ for the $\mathbb{Z}_p^2$-extension of K. Then one can view finite-order Hecke characters of K as p-adic Galois characters of $G_\infty = \mathrm{Gal}(K_\infty/K)$. Still working with a fixed finite-order character χ, Perrin-Riou, using work of Hida, constructs a p-adic L-function, which we will denote $L_p^{(1)}(f,\chi)$, on the space of $\mathbb{Z}_p^\times$-valued characters η of G_∞, satisfying an interpolation law of the form

$$L_p^{(1)}(f,\chi)(\eta) \doteq L(f,\chi\eta,1), \text{ for } \eta \text{ a finite order character.}$$

Here the explicit, nonzero constant hidden in the symbol $\doteq$ includes not only the Euler factor for $L(f,\chi\eta,1)$ at p as in the Kubota-Leopoldt case, but now also a p-adic period on the left-hand side, and a complex period on the right. The Heegner hypothesis and the interpolation law together force

$$L_p^{(1)}(f,\chi)(\mathbf{1}) = 0.$$

For a fixed choice ρ of continuous $\mathbb{Z}_p^\times$-valued character of G_∞, we have a "derivative of L_p in the ρ-direction," which we denote (hiding the role of ρ) by $L_p^{(1)'}$, given by the rule

$$L_p^{(1)'}(f,\chi) := \frac{d}{ds}L_p^{(1)}(f,\chi)(\rho^s)|_{s=0}.$$

Perrin-Riou considers the case of ρ the cyclotomic character. She also constructs a p-adic height pairing $\langle\ ,\ \rangle_p$, and obtains (Theorem 1.3 of [PR]):

Theorem I.2 (Perrin-Riou)**.** *One has*

$$L_p^{(1)'}(f,\chi)(\mathbf{1}) \doteq \langle Q_\chi, Q_\chi\rangle_p.$$

1.1.3 Higher-weight modular forms and the geometry of Kuga-Sato varieties

Generalizing the Gross-Zagier formula I.1 in a different direction, one can ask for a formula where now f is allowed to have even weight $k \geq 2$ (we still suppose that f has level N and trivial nebentypus). In this case, the center of the functional equation is now $s = \frac{k}{2}$, and the analytic theory of the local signs $\epsilon_v(f,\chi)$ does not change, i.e. the global sign still forces

$$L(f,\chi,\frac{k}{2}) = 0,$$

and so we expect to be able to express $L'(f,\chi,\frac{k}{2})$ in terms of invariants of some geometric object defined over a number field. Gross and Zagier conjectured in this case that the appropriate object would be a "Heegner cycle." To describe these cycles, we will work for simplicity with $X_1(N)$, so that there is a universal (generalized) elliptic curve $\mathcal{E} \to X_1(N)$. We will also ignore some complications that arise at cusps of $X_1(N)$. Write $r = k-2$.The Kuga-Sato variety W_r is defined to be the r-fold fiber product of $\mathcal{E}$ with itself over $X_1(N)$ (more accurately, due to problems above the cusps, it is a canonical desingularization thereof). Modular forms relate to the cohomology of W_r:

- To say that a holomorphic function f on the upper half plane is a weight k-modular form is exactly to say that the expression

$$\omega_f = f(\tau)d\tau \wedge dz_1 \wedge \ldots \wedge dz_r$$

gives a well-defined form in $H^{r+1}_{\mathrm{dR}}(W_r/\mathbb{C})$ (here τ is the "horizontal" coordinate on the upper half plane, and $z_1, \ldots, z_r$ are the standard coordinates on the fibers $\left(\frac{\mathbb{C}}{\langle 1,\tau\rangle}\right)^r$ of $W_r \to X_1(N)$.)

- Deligne's 2-dimensional Galois representation V_f attached to a normalized newform f is obtained as a subspace of $H^{r+1}_{\text{ét}}(W_r, \mathbb{Q}_\ell)$ stable under an action of the Hecke algebra.

In fact, Scholl has constructed a rank two *motive* M_f attached to a modular form f. Its de Rham realization is the space spanned by ω_f and its complex conjugate, and its étale realization is Deligne's representation V_f (see Theorem 1.2.4 of [Sch]). These considerations suggest that the correct generalization of the point on a curve on the geometric side of the Gross-Zagier formula in the higher-weight case should be *algebraic cycles* on W_r. In [GZ], Gross and Zagier proposed, for $r = 2s$, cycles fibered above a point corresponding to a fixed elliptic curve E with CM by the ring of integers of K of the form $X^s_{\sqrt{D}}$, where

$$X^s_{\sqrt{D}} = \Gamma_{\sqrt{D}} - (E \times \{0\}) - D(\{0\} \times E),$$

D is the discriminant of K, and $\Gamma_{\sqrt{D}}$ denotes the graph of the map $[\sqrt{D}] : E \to E$. Nekovář [Ne2], borrowing on work of Hida, generalized Perrin-Riou's p-adic L-function and p-adic height pairing to this setting, and proved a higher-weight analogue of Theorem I.2. Around the same time, Zhang [Zh1] established the complex analogue of Theorem I.1 in higher weight.

1.1.4 Higher-weight Hecke characters and a broken symmetry

Each of the formulas we have discussed so far suppose that χ is a finite order character of $\mathrm{Gal}(H/K)$. In [BDP], Bertolini, Darmon, and Prasanna study the case where this assumption is dropped, allowing χ to be a Hecke character of K of weight less than k.

Consider a Hecke character $\chi : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ of K with infinity type a pair of integers (ℓ_1, ℓ_2), which means that

$$\chi|_{(K\otimes\mathbb{R})^\times}(z) = z^{-\ell_1}\overline{z}^{-\ell_2}.$$

Then one has $\chi\overline{\chi} = \mathbf{N}^{\ell_1+\ell_2}$ where $\mathbf{N}$ denotes the norm character on ideals. To match the conventions of [BDP], we work with special values of the Rankin-Selberg L-function $L(f, \chi^{-1}, s)$ rather than $L(f, \chi, s)$.

The local sign $\epsilon_\infty(f, \chi^{-1})$ is no longer forced to be positive, but rather one has the dichotomy

$$\epsilon_\infty(f, \chi^{-1}) = \begin{cases} -1, & k > |\ell_1 - \ell_2| \\ 1, & k \leq |\ell_1 - \ell_2| \end{cases}$$

On the other hand, the local signs at finite places do not change (at least under the assumption that χ is an unramified Hecke character). It follows that the sign of the global functional equation for $L(f, \chi^{-1}, s)$ depends exclusively on the relationship between k and $\ell = |\ell_1 - \ell_2|$.

Because of this disparity, we say that χ is of type 1 if $k > \ell$ and of type 2 if $k \leq \ell$. One says that χ is *critical* for f if $s = 0$ is a critical value for $L(f, \chi^{-1}, s)$ in the sense of Deligne's conjecture on special values of L-functions, which translates into one of the following conditions holding:

- (The type 1 case): $1 \leq \ell_1, \ell_2 \leq k - 1$.
- (The first type 2 case): $\ell_1 \geq k$ and $\ell_2 \leq 0$.
- (The second type 2 case): $\ell_1 \leq 0$ and $\ell_2 \geq k$.

One says χ is *central critical* if in addition the central character of χ matches the Nebentypus of f (so that the same Rankin-Selberg L-function shows up on both sides of the functional equation) and the center of the functional equation for $L(f, \chi^{-1}, s)$ is $s = 0$. (This latter condition amounts to demanding that $\ell_1 + \ell_2 = k$.)

Bertolini, Darmon, and Prasanna then construct a p-adic L-function $L_p^{(2)}(f, \_)$ according to an interpolation law of the form

$$L_p^{(2)}(f, \_) \doteq L(f, \chi^{-1}, 0)$$

where the range of interpolation is the space of central critical characters in the first type 2 case. Note that these values are not forced a priori to vanish, and therefore neither are the values

$$L_p^{(2)}(f, \chi)$$

when χ is of type 1 (even though the classical L-function certainly vanishes at these points). In [BDP], therefore, the *values* of $L_p^{(2)}$ outside the range of interpolation are investigated, rather than the derivatives.

In the weight two case, their theorem can be stated as follows:

Theorem I.3 (BDP). *For χ central critical of type 1, write $\chi = \mathbf{N} \cdot \mu$ where μ is a finite order character. Then one has*

$$L_p^{(2)}(f, \chi) \doteq \log_p(Q_\mu)^2$$

Here $\log_p$ is the p-adic logarithm on $E(\mathbb{C}_p)$, which is the unique locally analytic primitive F on $E(\mathbb{C}_p)$ for the standard Néron differential ω on E such that

$$F : E(\mathbb{C}_p) \to \mathbb{C}_p$$

is a homomorphism of groups.

In the higher-weight case, which is also covered in [BDP], the relevant cycles are no longer Heegner cycles on Kuga-Sato varieties, the point being that the motive attached to a positive-weight unramified Hecke character of K lives naturally in a self-product of an elliptic curve E with CM by $\mathcal{O}_K$. One must therefore work with the enlarged variety $W_r \times E^r$.

The relevant *generalized Heegner cycles* are (r-fold products of) graphs of cyclic degree N-isogenies $E \to E'$, again supported in a fiber of $W_r \times E^r$ over a Heegner point of $X_1(N)$, modified by an idempotent designed to render them cohomologically trivial.

On the analytic side, in higher weight, the p-adic logarithm generalizes to a p-adic Abel-Jacobi map, which is defined precisely in Section 3.4 of [BDP] or Section 7.2.1 of this document. Write $S_k(\mathbb{Q}_p)$ for the space of cusp forms of level N and trivial Nebentypus, defined over $\mathbb{Q}_p$.

The p-adic Abel-Jacobi map may be viewed as a map from the space $\mathrm{CH}_0^{r+1}(W_r \times E^r)$ of cohomologically trivial codimension-$(r+1)$ cycles on $W_r \times E^r$ to the space

$$(S_k(\Gamma_0(N), \mathbb{Q}_p) \otimes \mathrm{Sym}^r\, H^1_{\mathrm{dR}}(E_{\mathbb{Q}_p}, \mathbb{Q}_p))^\vee$$

of functionals on a piece of the cohomology of the base change of $W_r \times E^r$ to $\mathbb{Q}_p$.

We mention two applications of Theorem I.3. Recent work of Skinner ([Sk]) establishes a *converse* to Gross-Zagier's work on the Birch-Swinnerton-Dyer conjecture: Skinner shows that if $Ш(E/\mathbb{Q})$ is finite and its Mordell-Weil rank is one, then $\mathrm{ord}_{s=1} L(E, s) = 1$. Skinner shows this by using an Iwasawa-theoretic argument of Xin Wan to establish that, for some choice of K, $L_p^{(2)}(f, \mathbf{1}) \neq 0$. It follows from Theorem I.3 that the logarithm of Q_χ is not zero, so Q_χ is non-torsion. It then follows from the original Gross-Zagier formula (Theorem I.1) that $L'(E, 1) \neq 0$. In another, purely geometric application, in [BDP2], Bertolini, Darmon, and Prasanna relate the p-adic Abel-Jacobi map to the *coniveau filtration* on the group $\mathrm{CH}^*_0(W_r \times E^r)$, and as a corollary construct examples where the generalized Heegner cycles are non-torsion in the Griffiths group of homologically trivial cycles modulo algebraically trivial cycles.

1.2 Removing the Heegner hypothesis

In this dissertation, we study the situation of [BDP] after dropping the Heegner hypothesis. Before describing the main results, it is necessary to review results of Zhang [Zh2] on the Gross-Zagier formula in this setting. We assume for the remainder of this section that N is prime to the discriminant of K.

1.2.1 Zhang's work on Shimura curves

Factor

$$N = N^+N^-,$$

where N^- is the product of the primes dividing N which remain inert in K. A prime v dividing N^- has local sign $\epsilon_v(f,\chi) = -1$, so in order to preserve the global sign of the functional equation, Zhang assumes an even number of primes divide N^- (else there is no Gross-Zagier-type result to be proved.)

For simplicity, assume also that N is squarefree. Write B for the quaternion algebra over $\mathbb{Q}$ with discriminant N^-. It is an indefinite quaternion algebra, which means there is an isomorphism $\iota_\infty : B \otimes \mathbb{R} \xrightarrow{\sim} M_2(\mathbb{R})$, because an even number of primes divide N^-. Fix a maximal order $\mathcal{O}_B$ of B. Let $\mathcal{O}_{B,N^+}$ denote an Eichler order of level N^+ in $\mathcal{O}_B$ (see section 2.3.1); write Γ_{B,N^+} for the group of norm one elements of this order.

The Riemann surface

$$\mathcal{H}/\Gamma_{B,N^+}$$

is a Shimura curve. So long as $N^- \neq 1$ (in which case this Riemann surface is just a modular curve), it is already compact.

Unsurprisingly, the Shimura curve has a moduli-theoretic interpretation; it is a coarse moduli space for principally polarized abelian surfaces A with an embedding $\mathcal{O}_B \hookrightarrow \mathrm{End}(A)$ and a certain type of level structure which depends on N^+ (defined in Section 2.5.1). Such a surface is often called a "false elliptic curve." To the point $\tau \in \mathcal{H}$, one attaches the complex torus

$$A_\tau = \frac{\mathbb{C}^2}{\iota_\infty(\mathcal{O}_B)\begin{pmatrix}\tau\\ \\ 1\end{pmatrix}}.$$

together with the natural embedding of $\mathcal{O}_B$ into its endomorphism ring (this torus comes with a canonical principal polarization).

The Shimura curve has a model X over $\mathbb{Q}$, which is a coarse moduli space for false elliptic curves with level structure. The analogues of Heegner points on modular curves are points in $X(H)$ first studied by Shimura. To define them, note that, because primes dividing N^- are inert in K and primes dividing N^+ are split in K, there is an embedding

$$\iota_K : K \hookrightarrow B$$

such that $\mathcal{O}_{B,N^+} \cap \iota_K(K) = \iota_K(\mathcal{O}_K)$. For such an embedding, there is a unique point $\tau \in \mathcal{H}$ fixed under the action of $\iota_\infty(\iota_K(K^\times)) \subset M_2(\mathbb{R})$. A fundamental theorem of Shimura shows that the image of τ under the uniformization map $\mathcal{H} \to \mathcal{H}/\Gamma_{B,N^+}$ is defined over H (see p. 58 of [Sh]). Write P for this Heegner point, and for a character $\chi \in \mathrm{Gal}(H/K)$, again write

$$P_\chi = \sum_{\sigma \in \mathrm{Gal}(H/K)} \chi^{-1}(\sigma) P^\sigma.$$

For simplicity, suppose $\chi \neq \mathbf{1}$, so that the divisor P_χ has degree zero and therefore it makes sense to write Q_χ for the image of P_χ in $\mathrm{Div}^0(E)$.

Using the height pairing $\langle\ ,\ \rangle$ on A_f, Zhang then shows the following (Theorem C of [Zh2]):

Theorem I.4 (Zhang)**.** *For f of weight 2, one has*

$$L(f, \chi, 1) \doteq \langle P_\chi, P_\chi \rangle$$

Critical in Zhang's proof is the *Jacquet-Langlands correspondence*, which, given the modular form f (which comes from the classical modular curve of level N), provides a modular form f_B for Γ_{B,N^+} with the same Hecke eigenvalues.

In related work, Disegni ([Di]) has recently established a p-adic analogue of Theorem I.4; that is to say, he has removed the Heegner hypothesis from Perrin-Riou's Theorem I.2, relating an extension of Perrin-Riou's p-adic L-function to p-adic heights of CM points on Shimura curves. Both Zhang and Disegni's work also treat the more complicated case of Hilbert modular forms of parallel weight 2, where $\mathbb{Q}$ is replaced by a totally real field (and B is replaced by a quaternion algebra over this field).

1.2.2 Main result and sketch of proof

In this document, we ask what happens to the p-adic L-function of [BDP] when the Heegner hypothesis is removed. We work with modular forms f of even weight $2r+2$. Writing $\mathcal{A}$ for the universal abelian surface over the Shimura curve X and $\mathcal{A}_r$ for its r-fold fiber product over X, the above discussion suggests that there should be a cycle on $\mathcal{A}_r \times A^r$, where A is a fixed false elliptic curve with CM by $\mathcal{O}_K$, whose image under the p-adic Abel-Jacobi map computes special values of a p-adic L-function.

Write F for a number field containing the ring class field of K mod N^+. There is an idempotent e in $B \otimes K$ selected in Section 2.7, and we state our results below in terms of the cohomology group $eH^1_{\mathrm{dR}}(A)$. (If the class number of K is odd, one can arrange for the fixed surface A to be a self-product $E \times E$ of CM elliptic curves, in which case $eH^1_{\mathrm{dR}}(A) = H^1_{\mathrm{dR}}(E)$.)

We then show the following:

Theorem I.5. *Suppose that f has even weight $k = 2r+2$, with $r \geq 0$, and χ is an unramified Hecke character of K of infinity type (ℓ_1, ℓ_2) with $\ell_1 + \ell_2 = k$ and $\ell_1, \ell_2 \geq 1$, so that $(\ell_1, \ell_2) = (k-1-j, 1+j)$ with $0 \leq j \leq 2r$. Then there is, for each $\mathfrak{a} \in \mathrm{Cl}(\mathcal{O}_K)$, an algebraic cycle $\Delta_r(\mathfrak{a})$ on*

$$X_r := \mathcal{A}_r \times A^r,$$

that is homologically trivial and defined over H, such that

$$\frac{L_p(f,\chi)}{\Omega_p^{4r-4j}} = E_p(f,\chi) \cdot \left\{ \sum_{[\mathfrak{a}] \in \mathrm{Cl}(\mathcal{O}_K)} \chi^{-1}(\mathfrak{a}) \cdot \mathrm{AJ}_p(\Delta_r(\mathfrak{a}))(\omega_{f_B} \wedge \omega_A^j \eta_A^{2r-j}) \right\}^2, \tag{1.1}$$

where

- AJ_p *is the p-adic Abel-Jacobi map, viewed as a map*

  $$CH_0^{2r+1}(X_{r/F_p}) \to (S_k(F_p) \otimes \mathrm{Sym}^{2r} eH^1_{dR}(A_{/F_p}))^\vee$$

  with F_p being the completion of F at the chosen prime above p and $S_k(F_p)$ the space of modular forms of weight k over F_p.

- *$E_p(f,\chi)$ is the Euler factor of $L(f, \chi^{-1}, s)$ evaluated at 0, and Ω_p is a p-adic period attached to A.*

- *f_B is the (suitably-normalized) Jacquet-Langlands lift of f to C, ω_{f_B} is the associated differential form on $\mathcal{A}_r$, and $\{\omega_A, \eta_A\}$ are a basis for $eH^1_{dR}(A/H)$,* *with ω_A holomorphic on $A(\mathbb{C})$ and η_A antiholomorphic on $A(\mathbb{C})$, normalized such that the cup product $\langle \omega_A, \eta_A \rangle = 1$.*

The main difficulty in proving Theorem I.5 is in making explicit computations with modular forms over Shimura curves. Because the groups Γ_{B,N^+} contain no translation matrices, a modular form on a Shimura curve does not have a q-expansion. While this makes the technical theory of Shimura curves easier in a few places (for instance, one has to worry about the integral structure of the fibers over the Kuga-Sato variety over the cusps in the classical case and not in the Shimura curve case), it also makes explicit calculation of Hecke operators and other operators on Shimura curves more challenging. To replace q-expansions, we use so-called Serre-Tate expansions, which are p-adic expansions of modular forms at CM points coming from the deformation theory of ordinary abelian varieties in characteristic p. We will give an explicit description (in Chapter V) for some differential operators and Hecke operators in terms of these expansions, following work of Brakočević [Br] and Mori [Mo], then apply those formulas to establish Theorem I.5.

Before explaining these formulas in a little more detail, we summarize the proof of the main theorem of [BDP]. In [BDP], a generalized Heegner cycle is built as described above, then modified by an algebraic projector, due to Scholl, designed to project the cohomology of the expanded Kuga-Sato variety $W_r \times E^r$ onto the subspace

$$S_{r+2}(\Gamma_1(N)) \otimes \mathrm{Sym}^r H^1(E).$$

Write Υ for their cycle. They compute the image of Υ under the p-adic Abel-Jacobi map in two steps. The first step is to relate this image to a "Coleman primitive" for the section of a line bundle on $X_1(N)$ attached to f. The second is to compute the Coleman primitive of f in terms of $\theta^{-1} f$, where $\theta = q\frac{d}{dq}$ is the Atkin-Serre p-adic differential operator which maps the space of p-adic modular forms of weight k to the space of p-adic modular forms of weight $k+2$. (In simple language, step one says that the p-adic Abel-Jacobi map is a p-adic integral, and step two says that one can evaluate this integral by undoing a derivative!) The values of θf coincide with the values of $\Theta_\infty f$ at CM points, where Θ_∞ denotes the Maass-Shimura operator

$$\frac{1}{2\pi i}\left(\frac{d}{d\tau}+\frac{k}{z-\overline{z}}\right).$$

The p-adic L-function is then constructed (and computed) using a Waldspurger-type result expressing values of the classical Rankin-Selberg L-function in terms of values of $\Theta_\infty^j f$ at CM points.

We now explain our proof. Our cycle starts life as the $2r$-fold power of the graph of an isogeny from the fixed elliptic curve E to some quotient E' (that is, as an isogeny $A^r \to (A')^r$ of "CM false elliptic curves"). It must be modified by an algebraic projector to be made homologically trivial. When $k > 2$, we use a projector originally constructed by Besser, which projects the cohomology of the expanded Kuga-Sato variety $\mathcal{A}_r \times A^r$ onto the subspace

$$M_{2r+2}(\Gamma_{B,N^+}) \otimes \operatorname{Sym}^{2r} H^1(A).$$

When $k = 2$, we instead use a projector that depends on f, namely, the standard projector, built from Hecke correspondences, cutting out the motive of f (see Section 7.3 for more details). (If χ induces a non-trivial character of $\mathrm{Gal}(H/K)$, we could take the same cycle Q_χ as above; the problem is that when χ gives the trivial character, this cycle is not homologically trivial. This differs from Zhang's solution to the similar problem that arises in [Zh2].)

We then evaluate the p-adic Abel-Jacobi map on the image of this cycle. The proof that the Abel-Jacobi map can be expressed in terms of a Coleman primitive closely follow [BDP]. However, the methods of [BDP] break down in inverting the differential operator θ, since that paper makes essential use of q-expansions. In fact, the Atkin-Serre formula for θ above does not even make sense over Shimura curves, but thanks to Katz, there is a moduli-theoretic interpretation of θ for PEL Shimura varieties over p-adic fields. To invert θ as an operator, we work in Serre-Tate coordinates, which are convergent expansions for modular forms on ordinary residue disks of X. We give formulas for $\theta^{-1}f$ on these expansions. As in [BDP], the p-adic L-function is then constructed (and computed) using a Waldspurger-type result.

Under the hypothesis that $p|N^-$ exactly once, Marc Masdeu [Ma] has proven a similar result by using a p-adic analytic uniformization of the corresponding Shimura curve, which has bad reduction at p. Such a uniformization, however, is not available in the case of good reduction. Conversely, our techniques rely heavily on the good reduction of the Shimura curve, and thus do not recover Masdeu's results.

1.2.3 Outline of this document

In Chapter II, which is expository in nature, we give a review of the theory of quaternion algebras over $\mathbb{Q}$ and the geometry of the associated Shimura curves. As explained above, Shimura curves are moduli spaces for false elliptic curves, and in the second half of this chapter, we also quote the results about false elliptic curves that we need. The reader who is already familiar with the basic theory of quaternion algebras may wish to skip the first half of this chapter. We then introduce the notation that we work with for most of the document, and outline the running assumptions of the document.

In Chapter III, we discuss the theory of modular forms on Shimura curves, defining the p-adic differential operator θ and its real analogue Θ_∞ discussed above, and we define some Hecke operators. The differential operators in question arise from the Gauss-Manin connection on the relative cohomology bundle of the universal false elliptic curve over the Shimura curve, and so we begin Chapter III with a review of this connection and the related Kodaira-Spencer map. We also use this interpretation of the differential operator to explain why the Atkin-Serre operator coincides with the Maass-Shimura operator (for values at CM points), a result originally due to Katz. Because ordinary false elliptic curves over p-adic fields have "canonical subgroups" killed by p (as do honest ordinary elliptic curves), the Hecke operator T_p splits into two pieces V and U. In the classical theory this can be seen already on the level of q-expansions. We state a commutation relation between the operator θ and the Hecke operators, although we defer its proof until we have built the necessary machinery.

In Chapters IV and V, we compute the differential operators of Chapter III in terms of expansions of modular forms at ordinary CM points. To do this, we first review Serre-Tate theory, which we will use to get an expansion for a modular form at an ordinary CM point that works as a substitute for a q-expansion. Serre-Tate theory gives us, on the one-hand, an explicit uniformizer in the ring of rigid analytic functions on a disk of the Shimura curve containing an ordinary CM point, and on the other hand, an explicit trivialization of the bundle of p-adic modular forms of weight k over this disk. We can thus use it to express any modular form locally "in coordinates," and we can ask what Hecke operators and differential operators do to these coordinates. Chapter IV reviews Serre-Tate theory in the general setting, and Chapter V performs the necessary calculations in the Shimura curve setting. The main results of these chapters are formulas for Hecke operators in these coordinates, and a proof that the Atkin-Serre operator is invertible on the space of "prime to p" p-adic modular forms. (A modular form is prime to p if it is fixed by the idempotent

p-adic Hecke operator $1 - UV$, as defined in Section 3.5. A classical p-adic modular form is "prime to p" if and only if all its Fourier coefficients of the form a_{np} vanish.)

In Chapter VI, we review residue theory on vector bundles with flat connections, and Coleman's p-adic methods for computing these residues. Although these methods are only used in Chapter VIII, the language of residue theory provides the motivation for the cycles constructed in Chapter VII.

In Chapter VII, we produce a homologically trivial cycle on a Kuga-Sato variety over the Shimura curve. The key input here in the higher weight case is work of Besser, which gives an algebraic projector P taking the cohomology of the Kuga-Sato variety to the space of quaternionic modular forms. In weight two, the cycle, in the case of non-trivial characters χ of $\mathrm{Gal}(H/K)$, can be taken as a weighted sum of Heegner points similar to Zhang's above; however, a uniform treatment which covers both the trivial and non-trivial character alike, and which is entirely due to Prasanna, is explained at the end of this chapter. In this chapter, we also define the p-adic Abel-Jacobi map.

In Chapter VIII, we follow [BDP] closely in interpreting the p-adic Abel-Jacobi map as a Coleman integral. We then use our formulas from Chapter V to compute the image of our cycle under the p-adic Abel-Jacobi map.

In Chapter IX, which is the analytic side of this document, we use these formulas and a Waldspurger-type result of Prasanna to build a p-adic L-function (the construction of which is originally due to Hida), then establish Theorem I.5. The argument, which is similar to that of [BDP], goes as follows: Prasanna's formula gives, for certain CM points $P_{\mathfrak{a}}$, the relation

$$L(f, \chi^{-1}, 0) \doteq \left| \sum_{\mathfrak{a} \in \mathrm{Cl}(\mathcal{O}_K)} \chi^{-1}(\mathfrak{a}) \mathbf{N}\mathfrak{a}^{-j} \cdot (\Theta_\infty^j f)(P_{\mathfrak{a}}) \right|^2 .$$

As mentioned earlier, $\Theta_\infty f$ and θf give the same value when evaluated at CM points. One can remove the absolute value signs from the Waldspurger formula (at the expense of introducing a new non-zero constant), and thus one can read the result as a p-adic formula expressed purely in terms of p-adic differential operators. This p-adic statement allows us to define the p-adic L-function on the space of type two Hecke characters, and gives a formula which self-evidently extends by continuity to the space of type one central critical Hecke characters.

CHAPTER II

Shimura curves

2.1 Initial choices

Fix an odd prime p, an isomorphism $\mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$, and compatible embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\mathbb{C}_p$. Fix also a newform f_{GL_2} of level N, with $p \nmid N$, of nebentypus ϵ_f, and of even weight $k = 2r + 2$ (with $r \geq 0$).

Let K be an imaginary quadratic field in which $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ splits. Factor $N = N^+N^-$, where primes dividing N^- are inert in K, and primes dividing N^+ are split or ramified in K. If a prime divides both N and the discriminant of K, assume also that it divides N exactly once (in other words, K satisfies the Heegner hypothesis with respect to the level N^+). Assume also that N^- is squarefree and divisible by an even number of primes.

Finally, choose a number field F containing the ring class field of K mod N^+ and the Hecke eigenvalues of F. This will be the field of definition of the arithmetic cycles constructed in this paper.

2.2 Review of quaternion algebras

This section reviews the general theory of quaternion algebras. The reader who is sufficiently acquainted with this theory to be comfortable with Section 2.5 should skip to Section 2.4. Let F be a field. A *quaternion algebra* B over F is a four-dimensional central simple algebra over F; thus it is an F-algebra with center exactly F and no non-trivial two-sided ideals. Two standard results on central simple algebras will be useful:

Theorem II.1 (Wedderburn's Theorem)**.** *Any central simple algebra is isomorphic to a matrix ring over a division algebra D (a* division algebra *is an F-algebra in which all nonzero elements are invertible).*

Theorem II.2 (The Noether-Skolem Theorem)**.** *Let A be a simple F-algebra and B a central simple F-algebra. If f, g are two homomorphisms from A to B, there is an element $x \in B^\times$ such that, for any $a \in A$, we have*

$$f(a) = x^{-1} g(a) x.$$

In particular, any automorphism of a central simple algebra is given by conjugation by an invertible element.

Now suppose that B is a quaternion algebra. By Wedderburn's theorem and a count of F-dimension, we see that B is either a division algebra or is isomorphic to $M_2(F)$.

If the characteristic of F is not 2, all quaternion algebras arise as follows: let $x, y \in F$. The symbol $(x, y|F)$ denotes the four-dimensional vector space on a basis $1, i, j, k$ together with the multiplication rules $i^2 = x$, $j^2 = y$, $ij = -ji = k$. For any field F, the quaternion algebra $(1, 1|F)$ is the trivial quaternion algebra $M_2(F)$. To see this, it suffices to find two anticommuting matrices M_i, M_j whose squares are the identity matrix; one choice is given by

$$M_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad\qquad M_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Further, if $x' = \alpha^2 x$ and $y' = \beta^2 y$, then $(x', y'|F) \xrightarrow{\sim} (x, y|F)$; the isomorphism is given by $i \mapsto \alpha i$, $j \mapsto \beta j$.

Definition II.3. A *pure quaternion* is an element of the subset of B consisting of 0, together with elements of $B \setminus F$ whose squares lie in F.

This subset, denoted B^{pure}, is actually a subspace: one checks that in the quaternion algebra $(a, b|F)$, the set of pure quaternions is the F-span of $\{i, j, k\}$.

Thus there is a decomposition of the underlying F-vector space of B

$$B = F \oplus B^{\mathrm{pure}}.$$

Consider the vector space automorphism $b \mapsto \bar{b}$ of B acting as 1 on F and -1 on B^{pure}, known as the "main involution." The *reduced trace* Tr is the additive map $B \to F$ given by

$$\mathrm{Tr}(b) = b + \bar{b},$$

and the *reduced norm* Nr is the map $B \to F$ given by

$$\mathrm{N}(b) = b\bar{b}$$

Note that the main involution satisfies

$$\overline{b_1 b_2} = \overline{b_2 b_1},$$

which shows that N lands in F (which is the subspace fixed by the main involution) and that N is multiplicative. One usually just speaks of the "trace" and "norm" of an element of B when there is no risk of confusing them with the trace or norm of that element acting on B by left-multiplication.

If $\beta \neq 0 \in B$ has norm zero, then it is a zero-divisor and B is not a division ring. Conversely, if β does not have norm zero, then $\beta^\sigma / N(\beta)$ is a multiplicative inverse for β; thus, a quaternion algebra B is a division algebra if and only if there are no elements of norm zero. This observation links quaternion algebras with quadratic forms: after writing $B = (a, b|F)$, there are any elements $x+yi+zj+wk$ of norm zero if and only if there are any solutions over F to the quadratic form in four variables given by

$$x^2 - ay^2 - bz^2 + abw^2 = 0.$$

We now describe the structure of quaternion algebras over local fields and $\mathbb{Q}$.

2.2.1 Quaternion Algebras over local fields

There are two quaternion algebras over $\mathbb{R}$: the two-by-two matrix ring $M_2(\mathbb{R})$ and Hamilton's quaternions $\mathbb{H} = (-1, -1|\mathbb{R})$.

Let L_π be a finite extension of $\mathbb{Q}_p$ with uniformizer π. Then, as in the real case, there are two isomorphism classes of quaternion algebras over L_π. They are the trivial quaternion algebra and $(\pi, u|L_\pi)$, where π is a uniformizer and $u \in \mathcal{O}_{L_\pi}{}^\times$ is a non-square. This latter algebra always splits (becomes isomorphic to a matrix algebra) over the unramified extension of degree 2 of L_π.

2.2.2 Quaternion Algebras over $\mathbb{Q}$

Now suppose that B is a quaternion algebra over $\mathbb{Q}$. The Hasse principle for the reduced norm form then implies that B is the trivial quaternion algebra if and only if $B \otimes L$ is the trivial quaternion algebra for each completion L of $\mathbb{Q}$.

One says that B *ramifies* at a place v of $\mathbb{Q}$ if $B \otimes \mathbb{Q}_v$ is a division algebra. The cardinality of the set of places (possibly including ∞) at which B ramifies is finite and even. The product Δ of the finite places at which B ramifies is called the *discriminant* of B. Given a squarefree positive integer Δ, there exists a quaternion algebra of discriminant Δ.

One calls a quaternion algebra *definite* if it ramifies at the Archimedean place and *indefinite* otherwise. Thus a quaternion algebra is definite if and only if its discriminant is divisible by an odd number of distinct primes. If $L/\mathbb{Q}$ is an imaginary quadratic field extension, then B splits over L if and only if every prime that ramifies in B remains inert in L.

2.3 Integral Structures in Quaternion Algebras

2.3.1 Integrality and Orders

Let $\mathcal{O}$ be a subring of L. An $\mathcal{O}$-*order* in a quaternion algebra is a subring which is finitely generated as an $\mathcal{O}$-module. Over global and (non-archimedean) local fields, we will take $\mathcal{O}$ the ring of integers, and hence say "order" without reference to $\mathcal{O}$. An order is *maximal* if it is not strictly contained in a larger order.

An element α in a quaternion algebra B is said to be $\mathcal{O}$-*integral* if it satisfies a monic polynomial with coefficients in $\mathcal{O}$; equivalently, if its "reduced minimal polynomial"

$$f(x) = (x-\alpha)(x-\alpha^\sigma) = x^2 - \mathrm{Tr}(\alpha)x + N(\alpha)$$

has coefficients in $\mathcal{O}$; equivalently, if $\mathcal{O}[\alpha]$ is a finitely generated $\mathcal{O}$-submodule of B.

It is clear from this last description that any order in B must consist exclusively of integral elements. In fact, any integral element is contained in a maximal order. If L is a non-archimedean local field, and B/L is the trivial quaternion algebra, then all maximal orders are conjugate to $M_2(\mathcal{O}_L)$. If B/L is the non-trivial quaternion algebra, then the set of all integral elements of B form a subring which is the unique maximal order. To give an order in a quaternion algebra over a global field, it is necessary and sufficient to give orders in each non-archimedean completion which are maximal at all but finitely many places.

An order in a quaternion algebra is an *Eichler order* if it is the intersection of two maximal orders. The "standard Eichler order of level π^n" in the trivial quaternion algebra over a local field L_π with uniformizer π is

$$\left\{ \begin{pmatrix} a & b \\ \pi^n c & d \end{pmatrix} | a, b, c, d \in \mathcal{O}_{L_\pi} \right\}.$$

For any Eichler order $\mathcal{O}$ in $M_2(L_\pi)$, there is some N such that $\mathcal{O}$ is conjugate to the standard Eichler order of level π^n.

If B is an indefinite quaternion algebra over $\mathbb{Q}$, then all maximal orders of B are conjugate. If N is an integer prime to the discriminant of B, and $\mathcal{O}_B$ is a fixed maximal order, then we say an order $\mathcal{O}_N \subset \mathcal{O}_B$ is an Eichler order of level N if its completion at each place p dividing N is conjugate to the standard Eichler order of level $p^{v_p(N)}$ and its completion at each place p dividing Δ is the maximal order. Any two Eichler orders of level N are conjugate.

2.4 Review of Shimura curves

2.4.1 Shimura Curves, Complex Analytically

Write $\mathcal{H}$ for the upper half plane. Let B be an *indefinite* quaternion algebra over $\mathbb{Q}$ with discriminant Δ, and fix an isomorphism

$$\iota : B \otimes \mathbb{R} \xrightarrow{\sim} M_2(\mathbb{R}).$$

In general, there is no canonical isomorphism (unless $B = M_2(\mathbb{Q})$), but nonetheless we will regard B as a $\mathbb{Q}$-subalgebra of $M_2(\mathbb{R})$ without explicitly writing ι.

Let $\mathcal{O}$ be an order in B, and pick a finite index subgroup Γ of the norm one elements in $\mathcal{O}$. Then one has a Riemann surface

$$Y_\Gamma = \mathcal{H}/\Gamma.$$

In the classical case of modular curves, where $B = M_2(\mathbb{Q})$ and $\Gamma \subset SL_2(\mathbb{Z})$, the curve Y_Γ is non-compact, and we write X_Γ for its canonical compactification. On the other hand, if B is a division algebra over $\mathbb{Q}$, then it is a consequence of the strong approximation theorem for B that Y_Γ is already compact, and so we write X_Γ for Y_Γ (see [We]). This compact Riemann surface is known as a *Shimura curve* (over $\mathbb{C}$). To describe rational (and integral) models of Shimura curves, we will reinterpret them as moduli spaces of "false elliptic curves" with level structure.

2.4.2 False Elliptic Curves

Let S be a scheme with $\Delta \in \mathcal{O}_S(S)^\times$. A *false elliptic curve* over S is an abelian surface A/S, together with an embedding $\mathcal{O} \hookrightarrow \mathrm{End}_S(A)$.

Over $\mathbb{C}$, there is a simple uniformization theorem for false elliptic curves. Recall that giving a principally polarized abelian surface over $\mathbb{C}$ is the same as giving a lattice Λ in $\mathbb{C}^2$, together with an $\mathbb{R}$-valued Hermitian form H on $\mathbb{C}^2$ with $\mathrm{Im} H(\Lambda, \Lambda) = \mathbb{Z}$. Given $\tau \in \mathcal{H}$, we can construct such data by taking

$$\Lambda = \mathcal{O} \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

and $H(v, w) = \langle v, w \rangle / \mathrm{Im}(\tau)$ (where $\langle\ ,\ \rangle$ is the standard Hermitian product on $\mathbb{C}^2$). Write A_τ for the principally polarized abelian variety $\mathbb{C}/\Lambda$ together with the natural embedding $\mathcal{O} \hookrightarrow \mathrm{End}(\mathbb{C}/\Lambda)$.

Theorem II.4 (Shimura). *Every false elliptic curve over $\mathbb{C}$ is of the form A_τ for some $\tau \in \mathcal{H}$.*

Proof. One very thorough reference (which proves much more) is Chapter 9 of [BL].

$\square$

If A and A' are false elliptic curves, a *false isogeny* between them is an isogeny commuting with the actions of $\mathcal{O}$. The *false degree* of a false isogeny is the square root of its degree. The *false endomorphism ring* of a false elliptic curve is the ring of false isogenies from the false elliptic curve to itself. We note especially that the non-central endomorphisms in $\mathcal{O}$ are *not* false endomorphisms. A false elliptic curve in characteristic zero said to have *complex multiplication* if its false endomorphism ring is strictly larger than $\mathbb{Z}$.

Let p be a prime at which B is split, and let A a false elliptic curve over a perfect field of characteristic p. The rank 4 p-divisible group of A gets an action of $B \otimes \mathbb{Z}_p = M_2(\mathbb{Z}_p)$, so is isomorphic to the square of a rank 2 p-divisible group. By the classification of rank 2 p-divisible groups, this latter group is either the p-divisible group of an ordinary elliptic curve or of a supersingular elliptic curve. One says the false elliptic curve is *ordinary* or *supersingular* respectively in these two cases.

2.5 The fixed Shimura curve

From now on, write B for the (necessarily indefinite) quaternion algebra over $\mathbb{Q}$ of discriminant N^-. Pick an embedding

$$\iota_\infty : B \hookrightarrow \mathbb{R}.$$

Since primes dividing N^- are inert in K, the quaternion algebra $B \otimes K$ is trivial, so K embeds in B. Pick a maximal order $\mathcal{O}_B$ in B, and trivializations $\iota_p : B \otimes \mathbb{Q}_\ell \to M_2(\mathbb{Q}_\ell)$ with $\iota_B(\mathcal{O}_B \otimes \mathbb{Z}_\ell) = M_2(\mathbb{Z}_\ell)$ for $\ell \nmid N^-$. In particular we get a trivialization

$$\iota_{N^+} : \mathcal{O}_B \otimes \mathbb{Z}/N^+\mathbb{Z} \to M_2(\mathbb{Z}/N^+\mathbb{Z}).$$

Fix $t \in \mathcal{O}_B$ with $t^2 = -\Delta$. Such a t lies in any Eichler order of level prime to the discriminant. There is an anti-involution of B defined by the rule

$$b \mapsto t^{-1}\overline{b}t,$$

which we write as $b \mapsto b^{\dagger}$. For any false elliptic curve over any base $\mathbb{Z}[\frac{1}{N}]$-scheme S, there is a unique principal polarization whose associated Rosati involution on $\mathrm{End}(A)$ restricts to $\dagger$ on $\mathcal{O}$ (this is a theorem of Milne over a field; over an arbitrary base $\mathbb{Z}\left[\frac{1}{\Delta}\right]$-scheme, see the discussion in Section 1 of [Bu]).

Write $\mathcal{O}_{B,N^+}$ for the standard Eichler order of level N^+ in $\mathcal{O}_B$; write Γ for the group of norm one units of $\mathcal{O}_B$ and Γ_{0,N^+} for that of $\mathcal{O}_{N^+}$. The group Γ_{0,N^+} admits a canonical map to $\frac{\mathbb{Z}}{N^+}$ sending

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \iota_{N^+}(B)$$

to d, and we call the kernel of this map Γ_{1,N^+}.

2.5.1 Arithmetic model

For S a $\mathbb{Z}[1/N]$-scheme and A/S a false elliptic curve a *full level* N^+ structure on A is an isomorphism of group schemes

$$A[N^+] \to \mathcal{O}_B \otimes (\mathbb{Z}/N\mathbb{Z})_{/S}$$

commuting with the action of $\mathcal{O}_B$. A *level structure of type* $V_1(N^+)$ is an inclusion

$$\mu_{N^+} \times \mu_{N^+} \to A[N^+]$$

commuting with the action of $\mathcal{O}_B$, where μ_{N^+} denotes the group scheme of N^+th roots of unity. Here $\mathcal{O}$ acts on the left hand side via the trivialization $\mathcal{O}_B \otimes \mathbb{Z}/N^+\mathbb{Z} = M_2(\mathbb{Z}/N^+\mathbb{Z})$. Then we have the following fundamental theorem ([Morita], Theorem 1):

Theorem II.5 (Morita). *For $N^+ > 3$, the moduli problem attaching to a $\mathbb{Z}[1/N]$-scheme S the set of isomorphism classes of false elliptic curves over S together with $V_1(N^+)$ level structure is representable by a smooth proper $\mathbb{Z}[1/N^+]$ scheme $\mathcal{C}$.*

2.6 CM points on Shimura curves

2.6.1 Complex uniformization of false elliptic curves and CM points

Because $\mathcal{O}_B$ is an order in B, there is a four-dimensional real torus

$$A_{\mathbb{R}} = \frac{B \otimes \mathbb{R}}{\mathcal{O}_B} = \frac{M_2(\mathbb{R})}{\iota_\infty(\mathcal{O}_B)}$$

endowed with endomorphisms of $\mathcal{O}_B$ via left-multiplication.

For $\tau \in \mathcal{H}$, write $J_\tau \in M_2(\mathbb{R})$ for the unique real matrix with

$$J_\tau \begin{pmatrix} \tau \\ 1 \end{pmatrix} = i \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Then right-multplication by J_τ puts a complex structure on $A_{\mathbb{R}}$ for which the endomorphisms coming from $\mathcal{O}_B$ are holomorphic.

Write A_τ for the corresponding false elliptic curve. Then there is an isomorphism of false elliptic curves

$$A_\tau \xrightarrow{\sim} \frac{\mathbb{C}^2}{\iota_\infty(\mathcal{O}_B)\begin{pmatrix} \tau \\ 1 \end{pmatrix}},$$

given by

$$M \mapsto M \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

There is an alternating form E on $B \otimes \mathbb{R}$ given by

$$E(x, y) = \mathrm{Tr}(ty\overline{x}).$$

This form gives a polarization on A_τ for which the Rosati involution on $\mathcal{O}_B$ is the Milne polarization $b \mapsto b^\dagger$.

Given an embedding

$$\iota : K \hookrightarrow B,$$

with $\iota(\mathcal{O}_K) \subset \mathcal{O}_B$, there is a unique $\tau \in \mathcal{H}$ with

$$\iota_\infty(\iota(K^\times)(\tau)) = \tau.$$

It follows that the additive map $K \to \mathbb{C}$ given by

$$\alpha \mapsto (j(\iota_\infty(\iota_K \alpha)), \tau),$$

is also multiplicative and hence an embedding of fields. The map ι is said to be *normalized* if the induced field embedding $K \hookrightarrow \mathbb{C}$ is the identity. The points $\tau \in \mathcal{H}$ for which such an embedding exists are called *CM points* (for B and $\mathcal{O}_K$). They are in bijective correspondence with normalized embeddings.

Write ι_τ for the normalized embedding $K \hookrightarrow B$ fixing τ. The group Γ acts by conjugation on the set of such embeddings, and this action satisfies

$$\gamma\iota_\tau = \iota_{\gamma\tau}.$$

Suppose that τ is a CM point. Then the false elliptic curve A_τ has false endomorphisms via right-multiplication by $\iota_\tau(\mathcal{O}_K)$ (these endomorphisms commute with the complex structure J_τ, and $\alpha \in \mathcal{O}_K$ induces the scalar $j(\iota_\infty(\iota_\tau\alpha), \tau) = \alpha$ on the tangent space of A_τ).

2.6.2 The action of $\mathrm{Cl}(K)$ and Shimura's reciprocity law

Suppose that τ is a CM point, and let $\mathfrak{a}$ be an (integral) ideal of $\mathcal{O}_K$. Then there is a left-ideal of $\mathcal{O}_B$ in B given by

$$\mathfrak{a}_B = \mathcal{O}_B(\iota_\tau(\mathfrak{a})).$$

Because B is an indefinite rational quaternion algebra, it has class number one and $\mathfrak{a}_B$ is *principal*, generated by some $\alpha \in B$.

Right-multiplication by α gives a false isogeny

$$A_\tau \to A_{\alpha^{-1}\tau}$$

with kernel $A_\tau[\mathfrak{a}]$, the subgroup of A_τ killed by all endomorphisms in the ideal $\mathfrak{a}$. If $(\mathfrak{a}, N) = 1$ and t is a level-N^+ structure on A_τ, this false isogeny induces a level-N^+ structure t_α on A_τ.

The image of $\alpha\tau$ under the uniformization map $\mathcal{H} \to \mathcal{H}/\Gamma_{0,N^+}$ does not depend on the choice of α. As a consequence, it makes sense to write

$$A_{\mathfrak{a}\star\tau}$$

for the corresponding false elliptic curve. Alternatively, one may view

$$A_{\mathfrak{a}\star\tau}$$

as the false elliptic curve

$$B \otimes \mathbb{R}/\mathfrak{a}_B$$

(with underlying complex structure J_τ). In these coordinates, the isogeny given above by right-multiplication by α^{-1} becomes the natural projection. Shimura's reciprocity law states that the point $\rho(\tau)$ is defined over H, and, moreover, for $\mathfrak{a} \in \mathrm{Cl}(K)$ one has

$$\rho(\tau)^{(\mathfrak{a}^{-1}, H/K)} = \rho(\mathfrak{a} \star \tau).$$

(Note that if one replaces $\mathfrak{a}$ by $\lambda\mathfrak{a}$ for some $\lambda \in K$, then the corresponding $\alpha \in B$ is replaced by $\alpha\iota_\tau(\lambda)$ and thus $A_{\mathfrak{a}\star\tau}$ does not change.) The set of isomorphism classes of CM false elliptic curves over H (or any field containing H) is thus a torsor for $\mathrm{Cl}(K)$ under the action $\star$.

2.7 Shimura curves and Kuga-Sato varieties

Write C for $\mathcal{C}_F$, $\mathcal{A}$ for the universal false elliptic curve over C, and $\mathcal{A}_r$ for the r-fold fiber product of $\mathcal{A}$ with itself over C (called the Kuga-Sato variety).

Fix forever an embedding $\iota : K \hookrightarrow B$ and write τ for its fixed point in $\mathcal{H}$ and A for the corresponding false elliptic curve over $F \supset H$. The surface A is ordinary at p, thanks to the assumption that p splits in K. Let

$$W_r = \mathcal{A}_r \times A^r.$$

This "enlarged Kuga-Sato variety" is the home of the arithmetic cycles which will be constructed in Chapter VII.

The fixed embedding ι_τ, together with a choice of basis for B as an $\iota_\tau(K)$ vector space, gives an identification

$$i_K : B \otimes K = M_2(K);$$

we may fix such an identification so that $i_K(b^\dagger) = i_K(b)^t$. Write

$$e = i_K^{-1}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right);$$

it is a non-trivial idempotent in $B \otimes K$ such that $e^\dagger = e$. Thanks to the chosen embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\mathbb{C}_p$, one can think of e as living in $B \otimes R$ for R any sub-K-algebra of either of these fields. In particular, this includes $R = \mathbb{Q}_p$, since p splits in K. Changing i_K if necessary, we may assume that $e \in \mathcal{O}_B \otimes \mathbb{Z}_p$.

Let $\mathcal{H}^1$ be the first relative de Rham cohomology bundle on C attached to the map $\mathcal{A} \to C$.

Write $\underline{\omega}$ for the bundle $e\Omega_{\mathcal{A}/C}$, and $\mathcal{L}_{2r}$ for $\mathrm{Sym}^{2r} e\mathcal{H}^1$. Note that $\mathcal{L}_{2r}$ is naturally a sub-bundle of the relative de Rham cohomology bundle $\mathcal{H}^{2r}(\mathcal{A}_r/C)$ of the rth Kuga-Sato variety over C.

The bundle $\mathcal{L}_{2r}$ is equipped with the (symmetric power of the) Gauss-Manin connection, the theory of which is reviewed in more detail in Chapter III. There is a Hodge exact sequence

$$0 \to \underline{\omega} \to \mathcal{L}_1 \to \underline{\omega}^{-1} \to 0.$$

When we write $\underline{\omega}^{-1}$ here, we are using the following fact: the standard identification of $R^1\pi_*\mathcal{O}_{\mathcal{A}}$ with the relative tangent bundle of the dual abelian scheme $\widehat{\mathcal{A}} \to C$, combined with the universal principal polarization $\widehat{\mathcal{A}} = \mathcal{A}$, gives rise to a (cotangent-tangent) pairing

$$\Omega_{\mathcal{A}/C} \times R^1\pi_*\mathcal{O}_{\mathcal{A}} \to \mathcal{O}_C,$$

and because e is fixed by the Rosati involution (which is just transposition on $B \otimes K = M_2(K)$), this pairing restricts to a perfect pairing $\underline{\omega} \otimes eR^1\pi_*\mathcal{O}_{\mathcal{A}} \to \mathcal{O}$.

Finally, there is a bundle $\mathcal{L}_{2r,2r}$ on C given by $\mathcal{L}_{2r,2r} = \mathcal{L}_{2r} \otimes \mathrm{Sym}^{2r}\, eH^1_{\mathrm{dR}}(A)$. The Gauss-Manin connection extends trivially to this bundle via the rule $\nabla(\alpha \otimes \beta) = \nabla\alpha \otimes \beta$.

2.8 The transfer

The *Jacquet-Langlands correspondence* asserts the existence of a holomorphic function f on the upper half plane with the following properties:

- f is a modular form for $\Gamma_{1,N^+} \subset B$ with Nebentypus ϵ_f for the action of Γ_{0,N^+}.
- f has weight k.
- For $(n, N^-) = 1$, f is an eigenform for the operator T_n with the same eigenvalue as f_{GL_2}.

These properties determine f as a holomorphic function on the upper half plane only up to a scalar multiple. However, one can normalize f further. The function f

gives rise canonically to a section of $\underline{\omega}_{\mathbb{C}}$, in the following manner: the universal false elliptic curve $\mathcal{A}_{\mathcal{H}}$ over $\mathcal{H}$ is the quotient of $\mathcal{H} \times \mathbb{C}^2$ by the action of $\mathcal{O}_B$ given by

$$b\left(\tau, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = \left(\tau, \iota_\infty(b)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right).$$

Because f is modular for Γ_{1,N^+}, the relative one-form

$$\omega_f = f(\tau) dz_1^{\otimes k} \in e\pi_* \Omega_{\mathcal{A}_{\mathcal{H}}}$$

for the universal false elliptic curve descends to a section of $\underline{\omega}_{\mathbb{C}}$. Because C and $\underline{\omega}$ both admit canonical models over $\mathcal{O}_F[1/N]$, and the Hecke eigenvalues of f lie in this ring, we may assume that our section is defined over this ring. Thus the choice of transfer is ambiguous up to multiplication by a unit in this ring.

2.9 Standard cohomology classes

Consider the Hodge exact sequence for A:

$$0 \to \Omega_{A/F} \to H^1_{\mathrm{dR}}(A/F) \to H^1(A, \mathcal{O}_A) \to 0.$$

Because A has CM, this sequence canonically splits, with $H^1(A, \mathcal{O}_A)$ identified as the subspace of $H^1_{\mathrm{dR}}(A/F)$ on which $\mathcal{O}_K$ acts via complex conjugation. In fact:

- over $\mathbb{C}$, this splitting coincides with the complex-analytic splitting of the Hodge sequence, i.e. the space $H^1(A_{\mathbb{C}}, \mathcal{O}_{A_{\mathbb{C}}})$ is identified with the subspace of $H^1_{\mathrm{dR}}(A/\mathbb{C})$ spanned by anti-holomorphic one-forms on $A(\mathbb{C})$.

- over any p-adic field L containing F, this splitting coincides with the "unit-root" splitting, i.e. after tensoring the Hodge sequence with L, the space $H^1(A_L, \mathcal{O}_{A_L})$ is identified with the subspace of $H^1_{\mathrm{dR}}(A/L)$ on which the semilinear Frobenius map ϕ acts via a unit.

To see these facts, note that, as is explained on p. 918 of [Pr], we may find a false isogeny $\phi : A \to E_1 \times E_2$, defined over H and with degree prime to p, where E_1 and E_2 are elliptic curves with CM by $\mathcal{O}_K$. For CM elliptic curves, the coincidence of the splittings of the Hodge exact sequence follows concretely from the observation that on the Weierstrass model $y^2 = 4x^3 + ax + b$, the differential

$$\frac{dx}{y}$$

lies in the subspace of $H^1_{\mathrm{dR}}(A/F)$ on which K acts via the identity embedding, the holomorphic subspace of $H^1_{\mathrm{dR}}(A/\mathbb{C})$, and the p-root subspace of $H^1_{\mathrm{dR}}(A/L)$, whereas the meromorphic differential form

$$x\frac{dx}{y}$$

lies in the subspace of $H^1_{\mathrm{dR}}(A/F)$ on which K acts via the conjugate embedding, the anti-holomorphic subspace of $H^1_{\mathrm{dR}}(A/\mathbb{C})$, and the unit-root subspace of $H^1_{\mathrm{dR}}(A/L)$.

Fix a non-vanishing differential $\omega \in eH^0(A, \Omega_A)$. This determines a class $\eta \in eH^1(A, \mathcal{O}_A)$ dual to ω_A under the Serre duality pairing. We will view ω and η as classes in $eH^1_{\mathrm{dR}}(A/F)$ (using the canonical splitting of the Hodge sequence for η).

2.10 Assumptions

The assumptions made on p, K, N, and f in the discussion above are:

- $(N, p) = 1$ (our methods depend on good reduction of the Shimura curve)
- p splits in K (the false elliptic curve A needs to have ordinary reduction at p)
- primes dividing N^+ are split or ramified in K; if ramified, they must divide N^+ at most once (to get Heegner points).
- $N^+ > 2$ (needed for $k > 2$ so that the moduli problem is representable)

- N^- is divisible by an even number of distinct primes (for B to exist)
- primes dividing N^- are inert in K.
- f has even weight. (for the construction of Besser's idempotent, later)

CHAPTER III

Modular forms and p-adic modular forms on Shimura curves

3.1 Modular forms on Shimura curves

3.1.1 Classical modular forms

There are several equivalent definitions of modular forms for Shimura curves. We will never need integrality conditions away from p, so we define them over algebras R over the localization $\mathcal{O}_{K,\mathfrak{p}}$ of $\mathcal{O}_K$ at $\mathfrak{p}$.

Extending the notation of the previous chapter a bit, if $\pi : A \to S$ is a (relative) false elliptic curve, write $\underline{\omega}_{A/S}$ for $e\pi_*\Omega_{A/S}$; in the particular case $\mathcal{A}_R \to \mathcal{C}_R$ of the universal false elliptic curve over an $\mathcal{O}_{K,\mathfrak{p}}$-algebra R, just write $\underline{\omega}_R$. The three definitions are:

Definition III.1. A modular form of weight k over R is a global section of $\underline{\omega}_R^{\otimes k}$.

Definition III.2. Let R_0 be an R-algebra. A *test triple* is a triple $(A/R_0, t, \omega)$ consisting of a false elliptic curve A over R_0, a $V_1(N^+)$ level structure t on A, and a nonvanishing global section of $\underline{\omega}_{A/R_0}$. Two test-triples $(A/R_0, t, \omega)$ and $(A'/R_0, t', \omega')$ over R_0 are isomorphic if there is an isomorphism $A \to A'$ taking t to t' and pulling ω' back to ω.

A modular form of level N^+ and weight f over R is a rule F that assigns to every isomorphism class of test triple $(A/R_0, t, \omega)$ over every R-algebra R_0, an element

$r \in R_0$, subject to the following axioms:

1. Compatibility with base change: If $f : R_0 \to R_0'$ is a map of R-algebras and A/R_0 is the base-change of A'/R_0' along f, one has

$$F(A, t, f^*\omega) = F(A', f(t), \omega)$$

2. The weight condition: For any $\lambda \in R_0$, one has $f(A/R_0, t, \lambda\omega) = \lambda^{-k} f(A/R_0, t, \omega)$.

Definition III.3. A *test pair* is a pair $(A/R_0, t)$ of a false elliptic curve $\pi : A \to \operatorname{Spec} R_0$ and a $V_1(N^+)$ level structure t. A modular form of weight k over R is a rule G that assigns a translation-invariant section of $\underline{\omega}_{A/R_0}^{\otimes k}$ to every isomorphism class of test pair $(A/R_0, t)$ over any R-algebra R_0, subject to the following base-change axiom: if $f : R_0 \to R_0'$ is a map of R-algebras and A is the base-change of A'/R_0' along f, one has

$$G(A, t) = f^* G(A', f(t)).$$

Given a modular form as in Definition III.3, we get a modular form as in Definition III.1 by taking the section given by the universal false elliptic curve with level structure $(\mathcal{A}_R/\mathcal{C}_R, t_r)$; this is an equivalence because $\mathcal{A}_R$ is universal. To go between Definitions III.3 and III.2, choose any translation-invariant global section ω and use the formula

$$G(A, t) = F(A, t, \omega)\omega^{\otimes k}$$

which is independent of this choice.

3.1.2 p-adic modular forms

Write L for the completion of the maximal unramified extension of $\mathbb{Q}_p$, W for the ring of integers of L, and k for the residue field $\overline{\mathbb{F}_p}$. By properness, there is a reduction map red : $C(L) = \mathcal{C}(L) \to \mathcal{C}(k)$. A *residue disk* D is a subset of $C(L)$ of the form

$$\{P \in C(L) | \text{red}(P) = x\}$$

for some fixed point $x \in C(k)$. A residue disk is not Zariski open, but is a (non-affinoid) open subset of C^{rig}, the rigid analytic space associated with C_L. Because $\mathcal{C}$ is smooth over W, each residue disk is conformal to the open unit disk in K (see section I.1 of [Co3]). The ring R_x of rigid functions on a residue disk D_x corresponding to a point $x \in C(k)$ is obtained from the ring $\mathcal{R}_x$ of functions on the formal completion of $\mathcal{C}$ at x by inverting p (Lemma 9.7 of [Kas]). Write C^{ord} for the *ordinary locus* of C^{rig}, the union of the residue disks above *ordinary* false elliptic curves.

If $\mathcal{V}$ is a vector bundle on C, we will sometimes write "a rigid-analytic section of $\mathcal{V}$" to mean a section of the associated vector bundle $\mathcal{V}^{\text{rig}}$ on some open subset of C^{rig}; similarly, when we write "locally-analytic section," we mean a section of the associated vector bundle $\mathcal{V}^{\text{la}}$ over some open subset of the topological space $C(L)$.

There are three equivalent definitions for a p-adic modular form of weight k over W for the Shimura curve C, analogous to Definitions III.1, III.2, and III.3, but working only with ordinary false elliptic curves over p-adically complete W-algebras.

Thus, a *p-adic modular form* for the Shimura curve is a rigid analytic section of the bundle $\underline{\omega}^{\otimes k}$ over C^{ord}. Equivalently, it is a rule F taking in triples (A, t, ω), where A is an *ordinary* false elliptic curve over some p-adically complete W-algebra R, t is level N-structure for A, and $\omega \in \underline{\omega}_{A/R}$, and returning an element of R, subject to compatibility with base change and the rule $F(A, t, r\omega) = r^{-k}F(A, t)$. Equivalently, it is a rule $\widetilde{F}$ taking in couples (A, t) and returning a section of $\underline{\omega}_{A/R}^{\otimes k}$, compatible with base change. A *locally analytic modular form* (over some open set in the p-adic topology), is a locally analytic section of the bundle $\underline{\omega}^{\otimes k}$.

3.2 The Gauss-Manin Connection

In this section, we review the construction of the Gauss-Manin connection ∇, which we will use, following Katz, to define differential operators on spaces of modular forms.

3.2.1 Complex-analytic construction

If S is a complex manifold and $\mathcal{V}$ is a vector bundle on $\mathcal{X}$, a *connection* on $\mathcal{V}$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_S} \Omega_S$$

satisfying the Leibniz rule

$$\nabla(fs) = f\nabla s + s \otimes df.$$

It is said to be a *flat* (or *integrable*) connection if the composition

$$\mathcal{V} \xrightarrow{\nabla} \mathcal{V} \otimes \Omega_S \xrightarrow{\nabla} \mathcal{V} \otimes \bigwedge^2 \Omega_S,$$

where ∇ is extended to $\mathcal{V} \otimes \Omega_S$ by the Leibniz rule, is the zero map (this condition is automatic if S is one-dimensional). The definitions above make sense in many other categories, including the category of T-schemes for some fixed base T, rigid analytic spaces, and the category of real manifolds, and we will use them in all of these categories.

Returning to complex manifolds, there is an equivalence of categories between the category of *d-dimensional local systems* on S, which are sheaves $\mathbb{V}$ that are locally isomorphic to the constant sheaf $\mathbb{C}^d$, and the category of vector bundles on S equipped with a flat connection. This equivalence attaches to a local system $\mathbb{V}$ the vector bundle $\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_X$, where the tensor product is taken over the constant sheaf $\mathbb{C}$, together with the connection $\mathrm{Id} \otimes d$. The local system may be recovered from the vector bundle as the sheaf of sections annihilated by ∇ (such a section is said to be *horizontal*).

If $\mathcal{V}(P)$ denotes the fiber of $\mathcal{V}$ at P, then a connection gives rise, for every path γ from a point $s \in S$ to a point $t \in S$, to an isomorphism $\phi_\gamma : \mathcal{V}(s) \to \mathcal{V}(t)$. The isomorphism ϕ_γ is called *parallel transport along* γ. If the connection is flat, then ϕ_γ depends only on the homotopy class of γ. Picking a basepoint s, the association $\gamma \mapsto \phi_\gamma$ gives a homomorphism

$$\pi_1(S, P) \to \operatorname{Aut} \mathcal{V}(P),$$

and this association gives rise to an equivalence between the category of d-dimensional local systems on S and the category of representations of $\pi_1(S, P)$ on d-dimensional complex vector spaces.

Let $\pi : \mathcal{X} \to S$ be a fibration of complex manifolds. If π is proper, the sheaf

$$R^i \pi_* \mathbb{C}$$

on S, where $\mathbb{C}$ denotes the constant sheaf on $\mathcal{X}$, is a local system whose fibers are canonically isomorphic to the cohomology of the fibers of $\mathcal{X}$ over S. We write ∇ for the associated flat connection on the vector bundle

$$R^i \pi_* \mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_S,$$

which is called the Gauss-Manin connection.

Parallel transport with respect to the Gauss-Manin may be described concretely as follows: for U a contractible open subset of S, $\pi^{-1}(U)$ deformation retracts onto a fiber, so the inclusion map $\mathcal{X}_s \to \pi^{-1}(U)$, for any $s \in S$, induces an isomorphism

$$H^i(\pi^{-1}(U), \mathbb{C}) \to H^i(\mathcal{X}_s, \mathbb{C})$$

and thus given a path γ between two points s and t, there is an isomorphism

$$\phi_\gamma : H^i(\mathcal{X}_s, \mathbb{C}) \to H^i(\mathcal{X}_t, \mathbb{C})$$

given by covering γ with finitely many contractible opens.

3.2.2 Algebraic construction

Now suppose that $\mathcal{X} \to S$ is a smooth morphism of schemes. If $\mathcal{X}$ and S are complete $\mathbb{C}$-schemes, the analytic vector bundle

$$R^i\pi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_S,$$

constructed above on the associated complex vector spaces must be algebraic, and the (relative) comparison theorem between Betti and algebraic de Rham cohomology identifies this bundle with the relative algebraic de Rham cohomology bundle, computed as the ith hyperderived functor $\mathbb{R}^i\pi_*(\mathcal{D}^\cdot)$, where $\mathcal{D}^\cdot$ denotes the relative de Rham complex

$$\mathcal{D}^\cdot = 0 \to \mathcal{O}_\mathcal{X} \to \Omega_{\mathcal{X}/S} \to \bigwedge^2 \Omega_{\mathcal{X}/S} \to \dots$$

(When $i = 1$ this is the bundle $\mathcal{H}^1$ of Section 2.7).

In [KO], Katz and Oda show how to recover ∇ algebraically; their construction works over an arbitrary base field. Setting

$$\mathrm{Fil}^i \bigwedge^q \Omega_\mathcal{X} = \mathrm{Image}\left((\pi^*\Omega_S)^i \otimes \Omega_\mathcal{X}^{q-i} \to \bigwedge^q \Omega_\mathcal{X} \right),$$

we get a decreasing filtration on the *absolute* de Rham complex

$$\mathcal{C}^\cdot = 0 \to \mathcal{O}_\mathcal{X} \to \Omega_\mathcal{X} \to \bigwedge^2 \Omega_\mathcal{X} \to \dots$$

of $\mathcal{X}$. The tautological exact sequence of chain complexes

$$0 \to \frac{\mathrm{Fil}^1 \mathcal{C}^\cdot}{\mathrm{Fil}^2 \mathcal{C}^\cdot} \to \frac{\mathcal{C}^\cdot}{\mathrm{Fil}^2 \mathcal{C}^\cdot} \to \frac{\mathcal{C}^\cdot}{\mathrm{Fil}^1 \mathcal{C}^\cdot} \to 0$$

gives rise to a long exact sequence in hypercohomology, in particular a connecting map

$$\mathbb{R}^i\pi_*\left(\frac{\mathcal{C}^\cdot}{\operatorname{Fil}^1\mathcal{C}^\cdot}\right) \to \mathbb{R}^{i+1}\pi_*\left(\frac{\operatorname{Fil}^1\mathcal{C}^\cdot}{\operatorname{Fil}^2\mathcal{C}^\cdot}\right).$$

The left-hand sheaf is, by smoothness of π, the same as $\mathcal{H}^i$. Another consequence of smoothness is that the natural map

$$\mathcal{C}^\cdot[-1] \otimes \pi^*(\Omega_{\mathcal{X}}) \to \frac{\operatorname{Fil}^1\mathcal{C}^\cdot}{\operatorname{Fil}^2\mathcal{C}^\cdot}$$

is an isomorphism. It follows that

$$\mathbb{R}^{i+1}\pi_*\frac{\operatorname{Fil}^1\mathcal{C}^\cdot}{\operatorname{Fil}^2\mathcal{C}^\cdot} = \mathbb{R}^i\pi_*(\mathcal{C}^\cdot \otimes \pi^*\Omega_{\mathcal{X}}),$$

but the projection formula gives a canonical isomorphism

$$\mathbb{R}^i\pi_*(\mathcal{C}^\cdot \otimes \pi^*\Omega_{\mathcal{X}}) \xrightarrow{\sim} \mathcal{H}^i \otimes \Omega_{\mathcal{X}},$$

so we get a map $\mathcal{H}^i \to \mathcal{H}^i \otimes \Omega_{\mathcal{X}}$. It is shown in [KO] that this map is a connection and that it coincides in the complex analytic category with the connection defined above.

3.3 The Kodaira-Spencer Map

We first recall the classical Kodaira-Spencer map. Let $\pi : \mathcal{X} \to S$ be a complex fiber bundle or a smooth morphism of T-schemes. We will write $\Omega_{\mathcal{X}}$ to mean the sheaf of holomorphic differentials in the former case or to mean $\Omega_{\mathcal{X}/T}$ in the latter, and similarly for Ω_S.

Let $s \in S$ and $v \in T_sS$. Thinking of v as a map from $\operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$ to $\mathcal{X}$, we get a deformation $\mathcal{X}_v$ of the fiber $\mathcal{X}_s$ of $\mathcal{X}$ above s by base change along $v \to S$, which corresponds to a class in

$$H^1(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s}).$$

This procedure gives rise to a homomorphism

$$T_sS \to H^1(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s})$$

which coincides with the connecting map attached to the short exact sequence

$$0 \to \mathcal{T}_{\mathcal{X}_s} \to \mathcal{T}_{\mathcal{X}} \to \pi^*\mathcal{T}_S|_{\mathcal{X}_s} \to 0$$

expressing the (trivial) normal bundle to $\mathcal{X}_s$ as the restriction to $\mathcal{X}_s$ of the pullback of the tangent bundle to S. If π is proper, then $H^1(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s})$ is just the fiber of $R^1\pi_*\mathcal{T}_{\mathcal{X}}$ at s, and these morphisms come from a morphism of vector bundles on S:

$$\mathcal{T}_S \to R^1\pi_*\mathcal{T}_{\mathcal{X}}.$$

For applications to abelian varieties, one often dualizes the construction of the deformation-theoretic Kodaira-Spencer map. Thus consider the short exact sequence of sheaves on $\mathcal{X}$

$$0 \to \pi^*\Omega_S \to \Omega_{\mathcal{X}} \to \Omega_{\mathcal{X}/S} \to 0,$$

which is exact on the left because π is smooth. Applying π_* gives a connecting homomorphism

$$\pi_*\Omega_{\mathcal{X}/S} \to R^1\pi_*(\pi^*\Omega_S) = R^1\pi_*\mathcal{O}_{\mathcal{X}} \otimes \Omega_S$$

the equality following from the adjunction formula. We will call this map the Kodaira-Spencer map.

If $\mathcal{X} \to S$ is an abelian scheme, there is a canonical isomorphism $R^1\pi_*\mathcal{X} = \mathcal{T}_{X^\vee/S}$, so one may think of the Kodaira-Spencer map as giving a map

$$\pi_*\Omega_{\mathcal{X}/S} \to \mathcal{T}_{\mathcal{X}^\vee/S} \otimes \Omega_S$$

or, which is the same, as a map

$$\mathrm{KS} : \pi_*\Omega_{\mathcal{X}/S} \otimes \pi_*\Omega_{\mathcal{X}^\vee/S} \to \Omega_S.$$

Remark III.4. The Kodaira-Spencer map is related to the Gauss-Manin connection ∇ as follows: The relative de Rham cohomology bundle $\mathcal{H}^1 := \mathbb{R}^1\pi_*(0 \to \mathcal{O}_X \to \Omega \to \ldots)$ sits in the Hodge exact sequence

$$0 \to \pi_*\Omega_{\mathcal{X}/S} \to \mathcal{H}^1 \to R^1\pi_*(\mathcal{O}_\mathcal{X}) \to 0.$$

The Kodaira-Spencer map is then recovered by composing the maps in this sequence with ∇; more precisely, it is the map

$$\pi_*\Omega_{\mathcal{X}/S} \to \mathcal{H}^1 \xrightarrow{\nabla} \mathcal{H}^1 \otimes \Omega_S \to R^1\pi_*(\mathcal{O}_\mathcal{X}) \otimes \Omega_S.$$

3.4 Katz's differential operators arising from the Hodge sequence.

The Gauss-Manin connection

$$\nabla : \mathcal{H}^1 \to \mathcal{H}^1 \otimes \Omega_C$$

on the relative de Rham cohomology bundle on the Shimura curve C is compatible with the anti-action of $\mathrm{End}_C(\mathcal{A})$ on $\mathcal{H}^1$ via the rule

$$\nabla \circ \phi = (\phi \otimes 1) \circ \nabla.$$

(See Proposition 2.2 of [Mo].) The Gauss-Manin connection thus naturally restricts to a connection on the bundle $\mathcal{L}_1$ and extends to the symmetric powers $\mathcal{L}_n$ of $\mathcal{L}_1$ by the Leibniz rule

$$\nabla(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = \sum_i v_1 \otimes \ldots \otimes \widehat{v_i} \otimes \ldots \otimes v_n \otimes \nabla(v_i).$$

(When $n = 0$, the connection is just $d : \mathcal{O}_C \to \Omega_C$.)

Using the universal principal polarization on $\mathcal{A}$, we think of the Kodaira-Spencer map as a map

$$\mathrm{KS} : \pi_*\Omega_{\mathcal{A}/C} \otimes \pi_*\Omega_{\mathcal{A}/C} \to \Omega_C.$$

By Theorem 2.5 of [Mo], this map becomes an isomorphism upon restricting to $\underline{\omega}\otimes\underline{\omega}$.

For each j we get a map $\tilde{\nabla} : \mathcal{L}_j \to \mathcal{L}_{j+2}$ by composing the maps

$$\mathcal{L}_n \xrightarrow{\nabla} \mathcal{L}_n \otimes \Omega_C \xrightarrow{\mathrm{id}\otimes\mathrm{KS}^{-1}} \mathcal{L}_n \otimes \underline{\omega}^{\otimes 2} \longrightarrow \mathcal{L}_n \otimes \mathcal{L}_2 \longrightarrow \mathcal{L}_{n+2}$$

where $\underline{\omega}^{\otimes 2} \to \mathcal{L}_2$ is just Sym^2 of the inclusion in the Hodge sequence.

Suppose we have a map $\Psi : \mathcal{H}^1 \to \pi_* \Omega_{\mathcal{A}/C}$ of vector bundles splitting the Hodge sequence. (In the cases considered below, the map Ψ will *not* be an algebraic morphism of vector bundles.) Write $\Psi^r : \mathcal{L}_n \to \underline{\omega}^{\otimes n}$ for the induced map on $\mathcal{L}_n$.

We then get a "differential operator" $\Theta_\psi : \underline{\omega}^{\otimes n} \to \underline{\omega}^{\otimes n+2}$ by the composition

$$\underline{\omega}^{\otimes n} \to \mathcal{L}_n \xrightarrow{\tilde{\nabla}} \mathcal{L}_{n+2} \xrightarrow{\Psi^r} \underline{\omega}^{\otimes n+2}. \tag{3.1}$$

When $n = 0$, this is just the inverse Kodaira-Spencer map (for any choice of splitting). Following Katz, we will apply this formalism in two different settings to attain differential operators on the spaces of smooth and p-adic modular forms.

3.4.1 The Maass-Shimura operator Θ_∞

A *real analytic modular form* (of weight k and level N^+) for B is an analytic function $f(z)$ satisfying the usual relation

$$f(\gamma z) = j(\gamma, z)^k f(z)$$

for $\gamma \in \Gamma_{B,N^+}$.

For $\mathcal{V}$ a vector bundle on $C_{\mathbb{C}}$, write $\mathcal{V}_{\mathrm{ra}}$ for the associated real analytic vector bundle on $C_{\mathbb{C}}$. We will describe a splitting $\Psi_\infty : \mathcal{H}^1_{\mathrm{ra}} \to \underline{\omega}_{\mathrm{ra}}$ of real analytic vector bundles over $C_{\mathbb{C}}$. The Dolbeault complex

$$\mathcal{O}_{C_{\mathbb{C}}} \to \mathcal{O}_{C_{\mathbb{C}},\mathrm{smooth}} \to \Omega^{0,1}_{C_{\mathbb{C}},\mathrm{smooth}} \to 0$$

is a resolution of $\mathcal{O}_{C_{\mathbb{C}}}$ ([Vo], Prop. 4.19), so identifies

$$\frac{\pi_* \Omega^{0,1}_{C_{\mathbb{C}},\text{smooth}}}{\overline{\partial} \pi_* \mathcal{O}_{C_{\mathbb{C}},\text{smooth}}} = R^1 \pi_* \mathcal{O}_{C_{\mathbb{C}}}.$$

Under this identification, the map

$$\mathcal{H}^1 \to R^1 \pi_* \mathcal{O}_{C_{\mathbb{C}}}$$

in the Hodge sequence is the natural projection. By Hodge theory, this projection splits on the level of real analytic bundles, because there is a unique *harmonic* section of $\mathcal{H}^1_{\text{ra}}$ in any $\overline{\partial}$-equivalence class of relative $(0,1)$ form. By definition

$$\Psi_\infty : \mathcal{H}^1_{\text{ra}} \to \underline{\omega}_{\text{ra}}$$

is the induced splitting of the Hodge sequence.

The differential operator coming from the splitting Ψ_∞ and the recipe in (3.1) is written Θ_∞ and called a Maass-Shimura operator. It sends real-analytic modular forms to real-analytic modular forms but does not preserve holomorphy. If we regard a modular form as a function f on the upper half plane and write the complex variable $z = x + iy$,then ([Mo], Proposition 2.9) the Maass-Shimura operator is given by the rule

$$\Theta_\infty(f) = \left(\frac{d}{dz} + \frac{k}{2iy} \right) f.$$

3.4.2 The Ramanujan-Atkin-Serre operator θ

There is a Frobenius morphism ϕ on the relative de Rham cohomology bundle $\mathcal{H}^*(\mathcal{A}^{\text{ord}}/C^{\text{ord}})$ over the ordinary locus C^{ord}, semilinear over L, inducing the usual Frobenius morphism ϕ on the fibers of this bundle. Moreover, there is a splitting Ψ_p of the Hodge sequence (of rigid vector bundles over the ordinary locus), where $R^1\pi_* \mathcal{O}_{\mathcal{A}^{\text{ord}}}$ is identified with the sub-bundle of $\mathcal{H}|_{C^{\text{ord}}}$ on which ϕ acts with unit eigenvalue (see Proposition 2.10 of [Mo]).

This splitting Ψ_p and the recipe in (3.1) give rise to a differential operator θ, taking p-adic modular forms of weight k to p-adic modular forms of weight $k+2$. If one regards the splitting Ψ_p as a map of bundles for the p-adic topology on $C^{\mathrm{ord}}(L)$, the same recipe gives rise to an operator on the space of locally analytic modular forms over the ordinary locus, also written as θ.

3.4.3 Coincidence of the operators at CM points

The following theorem is due to Shimura and Katz.

Theorem III.5. *If g is a modular form on C, and $P \in C(M)$ is a CM point for some number field M, then for any choice ω of translation-invariant differential on A, one has*

$$(\theta g)(P,\omega) = (\Theta_\infty g)(P,\omega)$$

in the sense that both numbers belong to M. (We are using the chosen embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\mathbb{C}_p$ in two ways: first, to get inclusions $C(M) \subset C(\mathbb{C}_p)$ and $C(M) \subset C(\mathbb{C})$, and second, to make sense of the equality.)

Proof. This is shown in many places; see e.g. Theorems 2.4.5 and 2.6.7 of [Ka3] in more generality or Proposition 2.11 of [Mo] for the Shimura curve case. The point is that at a CM point, the splitting of $H^1_{\mathrm{dR}}(A'/\mathbb{C})$ coming from Hodge theory and the splitting of $H^1_{\mathrm{dR}}(A'/\mathbb{C}_p)$ coming from Frobenius both come from the splitting of $H^1_{\mathrm{dR}}(A'/M)$ into the subspace where K acts via the identity and the subspace where K acts via conjugation, as discussed in Section 2.9. $\square$

3.5 Hecke operators and p-adic Hecke operators

Throughout this document, we follow the convention that Hecke operators act on the *right* on the space of modular forms, while differential operators act on the *left*.

This convention is unfortunate, as the differential operators and Hecke operators do not commute. The commutation relation is given by Proposition III.6 below.

For a prime ℓ, a false elliptic curve A over a field k of characteristic prime to ℓ has $\ell+1$ cyclic sub-$\mathcal{O}$-modules annihilated by ℓ. Write $C_0, \ldots, C_\ell$ for these subgroups and $\phi_i : A \to A/C_i$ for the associated to C_i. If

$$t : \mu_{N^+} \times \mu_{N^+}$$

is a $V_1(N^+)$ level structure on A, and $\ell \nmid N^+$, then $t_i = \phi_i \circ t$ is a $V_1(N^+)$ level structure on A/C_i. If ω is a one-form on A, then there is a unique one-form ω_i on A/C_i with $\phi_i^* \omega_i = \omega$.

The Hecke operator T_ℓ on the space of modular forms of weight k is defined by the averaging rule

$$F|_{T_\ell}(A, t, \omega) = \frac{1}{\ell} \sum_{i=0}^{\ell} F(A/C_i, t_i, \omega_i).$$

Note that the Hecke operators preserve the weight and level of a modular form, and also act on the larger space of p-adic modular forms.

Now suppose that A is a false elliptic curve with *ordinary* reduction over a p-adic field L. Then there is a unique p-torsion cyclic sub-$\mathcal{O}$-module C of A which reduces mod p to the kernel of the Frobenius morphism, called the *canonical subgroup* (this is Theorem 11.1 of [Kas], although in the case of ordinary reduction one may construct it more simply, following the discussion above the statement of that theorem). Order the p-torsion cyclic sub-$\mathcal{O}$-modules in such a fashion that C_0 is the canonical subgroup. If F is a p-adic modular form, we get another p-adic modular form $F|_V$ by the rule

$$F|_V(A, t, \omega) = F(A/C_0, \frac{1}{p} t_0, p\omega_0),$$

and a p-adic modular form $F|_U$ by the rule

$$F|_U(A,t,\omega) = \frac{1}{p}\sum_{i=1}^{p} F(A/C_i, t_i, \omega_i).$$

Writing $[p]$ for the operator on the space of modular forms given by

$$F|_{[p]}(A,t,\omega) = F(A, pt, \frac{1}{p}\omega)$$

one has

$$T_p = U + \frac{1}{p}[p]V.$$

One has $VU = \mathrm{Id}$. In particular, the operators UV and $VU - UV$ are idempotent.

Proposition III.6. *For any prime ℓ with $(\ell, N) = 1$, including the case $\ell = p$, one has*

$$(\theta f)|_{T_\ell} = \ell\theta(f|_{T_\ell}).$$

Proof. In the modular curve case, this is an easy consequence of the formula for T_ℓ on q-expansions (see paragraph 2.1 of [Se]). We give a proof in Serre-Tate coordinates in Section 5.5. This proof has the advantage of working for locally analytic modular forms which are rigid on residue disks (for which there is no q-expansion principle, even in the classical case). □

CHAPTER IV

Serre-Tate coordinates: general theory

This chapter and the next work under a notation scheme which conflicts with the one introduced in Chapter II. In these chapters (only), we work exclusively over the ring W of Witt vectors on $k = \overline{\mathbb{F}_p}$, writing L for its field of fractions. In these chapters, roman A always refers to abelian varieties over k and cursive $\mathcal{A}$ always to abelian schemes over other W-algebras (in particular, $\mathcal{A}$ no longer denotes the universal false elliptic curve and A no longer denotes the fixed false elliptic curve). Also, in these chapters, X refers to C_L and $\mathcal{X}$ refers to its integral model $\mathcal{C}_W$.

Fix a residue disk $D \subseteq X(L)$, the space of points reducing to a fixed ordinary false elliptic curve A/k with level structure t. The ring of rigid analytic functions on D is obtained from the ring of functions on the formal completion of $\mathcal{X}$ at the point corresponding to A on the special fiber by inverting p. There is a canonical formal uniformizer for this ring, coming from Serre-Tate theory, which we will use to give explicit formulas for the operators U, V, and of the preceding chapter. In the next chapter, we will give those formulas. Before explaining this, we review the basics of Serre-Tate theory.

4.1 Serre-Tate coordinates

This section recalls the results of Serre, Tate, and Katz on deformations of ordinary abelian varieties in characteristic p; for a detailed exposition with proofs, see [Ka2]. Fix a g-dimensional *ordinary* abelian variety A over k. Write A^t for the dual abelian variety. If R is an Artin local ring with residue field k, then a *deformation* of A to R is an abelian scheme $\mathcal{A}$ over R together with an identification $\mathcal{A} \times k \xrightarrow{\sim} A$.

Write T_pA and T_pA^t for the "physical" Tate modules of A and A^t, i.e.

$$T_pA = \varprojlim A[p^n](k).$$

They are free $\mathbb{Z}_p$-modules of rank g (by ordinarity).

Whenever we refer to the Weil pairing on A, we mean the scheme-theoretic Weil pairing, normalized as in Section 5 of [Ka2] (the classical Weil pairing is trivial in characteristic p)[1]). It is a non-degenerate alternating pairing of k group-schemes

$$e_{p^n} : A[p^n] \times A^t[p^n] \to \mu_{p^n}$$

restricting to a perfect pairing

$$\widehat{A}[p^n] \times A^t[p^n](k) \to \mu_{p^n},$$

compatible with the maps $p : A[p^n] \to A[p^{n-1}]$. (Here $\widehat{A}$ is the formal completion of A at the origin.)

Let $\mathcal{A}$ be a deformation of A to R. The formal group $\widehat{\mathcal{A}}$ represents the functor

$$\{\text{Artin local } R-\text{algebras with residue field } k\} \to \{\text{Groups}\}$$

given by

$$\widehat{\mathcal{A}}(B) = \ker(\mathcal{A}(B) \to \mathcal{A}(k)).$$

[1]The scheme-theoretic Weil pairing is due to Oda. The standard reference is [KM]; the pairing there is the inverse of Oda's pairing and the pairing in [Ka2]. The normalization of the Weil pairing does not affect any of the formulas in this document.

Then there is a pairing

$$q_{\mathcal{A}} : T_pA \times T_pA^t \to \widehat{\mathbb{G}_m}(R)$$

given by the following rule: given $P_n \in A[p^n](k)$, and $Q_n \in A^t[p^n](k)$, pick a lift $\widetilde{P_n}$ of P_n to $\mathcal{A}(R)$. Consider $p^n\widetilde{P_n} \in \widehat{\mathcal{A}}(R) \subseteq \mathcal{A}(R)$. If n is large enough that $(1+\mathfrak{m}_R)^{p^n} = 1$, then $\widehat{\mathcal{A}}(R)$ is killed by p^n. Thus it makes sense to compute the Weil pairing $e_{p^n}(p^n\widetilde{P_n}, Q_n^t)$, which is an element of $\mu_{p^n}(R)$, which for n large coincides with $\widehat{\mathbb{G}_m}(R)$. These elements are compatible for large n, which gives the desired map $q_{\mathcal{A}}$.

The Serre-Tate theorem asserts that this construction gives a bijection

$$\{\text{Isomorphism classes of deformations of } A \text{ to } R\} = \mathrm{Hom}_{\mathbb{Z}_p}(T_pA \otimes T_pA^t, \widehat{\mathbb{G}_m}(R))\}.$$

In particular, the left hand side, which is a priori only a set, gains the structure of a $\mathbb{Z}_p$-module. Furthermore, this correspondence is functorial in R. More precisely, writing $\mathcal{M}$ for the functor from the category of Artin local rings to the category of sets given by

$$\mathcal{M}(R) = \{\text{Isomorphism classes of deformations of } A \text{ to } R\},$$

we have

$$\mathcal{M} = \mathrm{Hom}_{\mathbb{Z}_p}(T_pA \otimes T_pA^t, \widehat{\mathbb{G}_m}).$$

Because these equivalences are compatible with inverse limits, we may replace the category of Artin local rings with the category of complete local rings in all of the preceding discussion (although the recipe for computing the pairing $q_{\mathcal{A}}$ only makes sense over Artin local rings). The following proposition gives us a helpful shortcut in calculating Serre-Tate coordinates:

Proposition IV.1. *If* $\mathbf{f} : \mathcal{A} \to \mathcal{A}'$ *is a map lifting some map* $f : A \to A'$, $P \in T_pA$, *and* $Q^t \in T_pA'^t$, *then one has*

$$q_{\mathcal{A}}(P, f^t(Q^t)) = q_{\mathcal{A}'}(f(P), Q^t).$$

Now, the functor $\mathcal{M}$ parameterizing all deformations of our fixed ordinary abelian variety A is a formal scheme $\mathrm{Spf}\,\mathcal{R}$ equipped with a universal formal abelian scheme $\pi : \widehat{\mathcal{A}} \to \mathrm{Spf}\,\mathcal{R}$. Write $\widehat{\mathcal{A}}$ for the completion of the universal abelian scheme above x.

By the above, there is then a bilinear pairing

$$q_{\widehat{\mathcal{A}}} : T_pA \otimes T_pA^t \to 1 + \mathfrak{m}_{\mathcal{R}},$$

and by universality, given any deformation $\mathcal{A}$ of A over any complete W-algebra R with residue field k, we have a map $\mathcal{R} \to R$ making the following triangle commute:

$$\begin{tikzcd} & T_pA \otimes T_pA^t \arrow[dl, "q_{\widehat{\mathcal{A}}}"'] \arrow[dr, "q_{\mathcal{A}}"] & \\ 1 + \mathfrak{m}_{\mathcal{R}} \arrow[rr] & & 1 + \mathfrak{m}_R \end{tikzcd}$$

The ring $\mathcal{R}$ is the completion of the W-algebra generated by the functions $q(P, Q^t) - 1$, as P and Q^t range over T_pA and T_pA^t, respectively, subject to the relations generated by the bilinearity of the pairing q. In particular, suppose that we pick bases $\{P_1, \ldots, P_g\}$ and $\{Q_1^t, \ldots, Q_g^t\}$ of T_pA and T_pA^t. Then we have g^2 elements $q_{ij} = q(\widehat{\mathcal{A}}, P_i, Q_j^t) \in \mathcal{R}$, and, writing

$$T_{ij} = q_{ij} - 1$$

we get a ring isomorphism

$$\mathcal{R} = W[[T_{ij}]].$$

4.2 Katz's computation of ∇

Serre-Tate coordinates give us a canonical way to compute the Gauss-Manin connection on the formal relative de Rham cohomology bundle $\widehat{\mathcal{H}} = \mathbb{R}^1\pi_*(\Omega^{\cdot}_{\mathcal{A}/\mathcal{R}})$ on residue disks over ordinary points. In this formal setting, there are line bundles $\pi_*\Omega_{\widehat{\mathcal{A}}/\mathcal{M}}$ and $R^1\pi_*\mathcal{O}_{\widehat{\mathcal{A}}} = \mathrm{Lie}(\widehat{\mathcal{A}}^t/\mathcal{R})$, sitting in the usual Hodge exact sequence

$$0 \to \pi_*\Omega_{\widehat{\mathcal{A}}/\mathcal{M}} \to \widehat{\mathcal{H}} \to \mathrm{Lie}(\widehat{\mathcal{A}}^t/\mathcal{R}) \to 0.$$

There is likewise a Gauss-Manin connection $\nabla : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}} \otimes \Omega_{\mathrm{Spf}\,\mathcal{R}}$. We will abuse notation and not distinguish these line bundles from $\mathcal{R}$-modules, starting right now:

Lemma IV.2 (Katz)**.** *There are canonical isomorphisms*

$$(\star) \qquad T_pA^t \otimes \mathcal{R} \xrightarrow{\sim} \pi_*\Omega_{\widehat{\mathcal{A}}/\mathcal{M}}$$

and

$$(\star\star) \qquad \mathrm{Hom}(T_pA, \mathbb{Z}_p) \otimes \mathcal{R} \xrightarrow{\sim} \mathrm{Lie}(\widehat{\mathcal{A}}^t/\mathcal{R}).,$$

such that:

- *the $\mathcal{R}$-semilinear Frobenius endomorphism Φ of $\widehat{\mathcal{H}}$ acts via multiplication by p on T_pA^t under the identification $(\star)$.*
- *the sub-$\mathbb{Z}_p$-module of $\widehat{\mathcal{H}}$ on which Φ acts via the identity maps isomorphically to* $\mathrm{Hom}(T_pA, \mathbb{Z}_p)$ *under the map $\widehat{\mathcal{H}} \to Lie(\widehat{\mathcal{A}}^t/\mathcal{R})$ and the identification $(\star\star)$.*

Proof. We just recall the construction of the isomorphisms here. For the computation of the Frobenius action, see Lemma 4.2.1 of [Ka2]. The Weil pairing gives an isomorphism

$$T_pA^t = \mathrm{Hom}_{\mathbb{Z}_p}(\widehat{\mathcal{A}}, \widehat{\mathbb{G}_m}),$$

so given $P \in T_pA^t$, one gets a differential on $\widehat{\mathcal{A}}$ by pulling back $\frac{dT}{T}$.

Dually (and swapping the roles of A and A^t), the Weil pairing gives rise to an isomorphism

$$\widehat{\mathcal{A}^t} = \mathrm{Hom}(T_pA, \widehat{\mathbb{G}_m}).$$

Applying "Lie" to both sides gives the second result, since for any R one has

$$\ker\left(\mathrm{Hom}(T_pA, \widehat{\mathbb{G}_m}(R[\epsilon]) \to \mathrm{Hom}(T_pA, \widehat{\mathbb{G}_m}(R))\right) = \mathrm{Hom}(T_pA, 1+\epsilon R) \approx \mathrm{Hom}(T_pA, R).$$

□

For $Q^t \in T_pA^t$, write $\widehat{\omega}_{Q^t}$ for the differential form coming from the lemma. If $\phi \in \mathrm{Hom}(T_pA, \mathbb{Z}_p)$, write $\widehat{\eta}_\phi$ for the image in $\widehat{\mathcal{H}}$ of the vector field attached to ϕ by the lemma under the Frobenius splitting of the Hodge sequence. Fix a basis $P_1, \ldots, P_g$ of T_pA, and write $P_i^\vee$ for the dual basis of $\mathrm{Hom}(T_pA, \mathbb{Z}_p)$. Then the Gauss-Manin connection on $\widehat{\mathcal{H}}$ is computed as follows:

Theorem IV.3 (Katz)**.** *One has*

$$\nabla \widehat{\eta}_{P_i^\vee} = 0$$

and, for any $Q^t \in T_pA^t$, one has

$$\nabla \widehat{\omega}_{Q^t} = \sum_i \widehat{\eta}_{P_i^\vee} \otimes \mathrm{dlog}\, q(P_i, Q^t).$$

Proof. This is "version quat." of the Main Theorem of [Ka2], as is stated in section 4.1 of that paper. □

Thus the horizontal sections of the Gauss-Manin connection are generated by expressions of the form

$$\widehat{\omega}_{Q^t} - \sum_i \log q(P_i, Q^t) \widehat{\eta}_{P_i^\vee}.$$

The following observation, which is Lemma 3.5.1 of [Ka2], is a simple consequence of the construction of the isomorphism in Lemma IV.2:

Lemma IV.4. *Given a map $f : A \to B$ of ordinary abelian varieties in characteristic p that deforms to a map of universal formal abelian varieties $\mathbf{f} : \widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$, and $P^t \in T_p B^t$, one has*

$$\mathbf{f}^* \widehat{\omega}_{P^t} = \widehat{\omega}_{f^*(P^t)}.$$

CHAPTER V

Serre-Tate coordinates: the case of Shimura curves

Recall that this chapter and the previous one use different notation from the rest of the document; refer to the first paragraph of the previous chapter for a review.

5.1 Serre-Tate coordinates for Shimura curves

Now assume that A is a false elliptic curve over k. In this case, we have a subfunctor $\mathcal{M}^{\text{false}}$ of $\mathcal{M} = \operatorname{Spf} \mathcal{R}$ taking an Artin local ring R with residue field k to the set of "false deformations" of A to R, where a false deformation is a deformation $\mathcal{A}$ of A to R together with an embedding $\mathcal{O} \to \operatorname{End}_{\mathcal{R}}(\mathcal{A})$ deforming the given embedding $\mathcal{O} \to \operatorname{End}_k(A)$ (deformations of the extra endomorphisms, if they exist, are unique – see Theorem 2.4 of [Ka2]).

Proposition V.1. *The subfunctor $\mathcal{M}^{false}$ of $\mathcal{M}$ is a formal subgroup-scheme. The ring of formal functions $\mathcal{R}^{false}$ on $\mathcal{M}^{false}$ is the quotient of $\mathcal{R}$ by the closed ideal generated by the relations*

$$q(bP, Q^t) = q(P, b^\dagger Q^t)$$

for $b \in \mathcal{O}$.

Proof. The relations $q(bP, Q^t) = q(P, b^\dagger Q^t)$ are a classical property of the Weil pairing. To see that these are the only relations (which is the remaining content of the proposition), see Proposition 3.3 of [Mo]. $\square$

Restricting the Hodge sequence of vector bundles on $\mathcal{M}$ to $\mathcal{M}^{\text{false}}$ recovers the Hodge sequence for the universal false elliptic curve $\mathcal{A}_{\mathcal{M}^{\text{false}}}/\mathcal{M}^{\text{false}}$, and for a class $\eta \in \mathcal{H}^1(\mathcal{M}/\mathcal{R})$ one has

$$(\nabla\eta)|_{\mathcal{M}^{\text{false}}} = \nabla(\eta|_{\mathcal{M}^{\text{false}}})$$

by the functoriality of the construction of the Gauss-Manin connection.

Lemma V.2. *The Katz isomorphisms of Lemma IV.2 are* †*-equivariant for the action of* $\mathrm{End}(\mathcal{A})$*; that is, one has*

$$\widehat{\omega}_{bQ^t}|_{\mathcal{M}^{false}} = [b^\dagger]^*\widehat{\omega}_{Q^t}|_{\mathcal{M}^{false}}$$

for $b \in \mathrm{End}(\mathcal{A})$*, and similarly, for* $\phi \in \mathrm{Hom}(T_pA, \mathbb{Z}_p)$ *one has*

$$\widehat{\eta}_{b^*\phi} = (b^\dagger)^*\widehat{\eta}_\phi.$$

Proof. By definition,

$$\widehat{\omega}_{bQ^t} = \psi_b^*\frac{dT}{T},$$

where ψ_b is the Weil-pairing map

$$(bQ_t, \_\_).$$

Writing ψ for the map $(Q_t, \_\_)$, the map ψ_b decomposes as $\psi_b = \psi \circ b^\dagger$, and the first result follows.

The second argument is similar. The construction of the isomorphism says to think of $b\phi$ as a map to $1+\epsilon R$, and then write this map as Weil-pairing against some $\xi \in \mathrm{Lie}(\widehat{A})$. But now ϕ will be Weil-pairing against $b^\dagger\xi$.

$\square$

Pick a basis $\{P_1, P_2\}$ for T_pA such that $eP_1 = P_1$ and $eP_2 = 0$; denote by P_1^t, P_2^t the images of P_1 and P_2 in T_pA^t under the canonical principal polarization. These choices give rise to sections of the formal bundles $\widehat{\pi_*\mathcal{A}/\mathcal{X}}$ and $\widehat{\mathrm{Lie}(\mathcal{A}^t/\mathcal{X})}$ via the Katz isomorphisms of Lemma IV.2, which we denote $\widehat{\omega}_{P_i^t}$ and $\widehat{\eta}_{P_i^\vee}$. To compute the Gauss-Manin connection in the situation that we desire, we will compute it on these sections over the formal 4-fold $\mathcal{M}$, restrict to $\mathcal{M}^{\text{false}}$, then apply e.

Because the Katz isomorphisms in Lemma IV.2 are equivariant for the action of $\mathrm{End}(\mathcal{A})$, one has

$$e^* \begin{pmatrix} \widehat{\omega}_{P_1^t}|_{\mathcal{M}^{\text{false}}} \\ \widehat{\omega}_{P_2^t}|_{\mathcal{M}^{\text{false}}} \\ \widehat{\eta}_{P_1^\vee}|_{\mathcal{M}^{\text{false}}} \\ \widehat{\eta}_{P_2^\vee}|_{\mathcal{M}^{\text{false}}} \end{pmatrix} = \begin{pmatrix} \widehat{\omega}_{P_1^t}|_{\mathcal{M}^{\text{false}}} \\ 0 \\ \widehat{\eta}_{P_1^\vee}|_{\mathcal{M}^{\text{false}}} \\ 0 \end{pmatrix}$$

For the remainder of this document, abbreviate $e^*\widehat{\omega}_{P_1}$ as $\widehat{\omega}$ and $e^*\widehat{\eta}_{P_1}$ as $\widehat{\eta}$. The choice of basis of T_pA gives us functions $q(P_i, P_j^t)$ for $i, j = 1, 2$; abbreviate the particular function $q(P_1, P_1^t)$ as just q.

Theorem V.3. *One has*

$$\nabla\widehat{\omega} = \widehat{\eta} \otimes \mathrm{dlog}\, q$$

$$\nabla\widehat{\eta} = 0.$$

Proof. This follows from theorem IV.3 and the rule $\nabla \circ e = (1 \otimes e) \circ \nabla$. □

As a corollary the theorem gives

$$\mathrm{KS}(\widehat{\omega}^{\otimes 2}) = \mathrm{dlog}\, q. \tag{5.1}$$

5.2 The operator θ in coordinates

Recall that θ is defined by stringing together the maps

$$\underline{\omega}^r \to \mathcal{L}_r \xrightarrow{\nabla} \mathcal{L}_r \otimes \Omega \xrightarrow{\Psi_p} \widehat{\underline{\omega}}^r \otimes \Omega \xrightarrow{KS^{-1}} \underline{\omega}^{r+2}$$

To compute the effect of this map on a section ω of the bundle $\mathcal{L}_r$ over the ordinary locus C^{ord}, we compute separately in each residue disk. Thus, along a fixed residue disk D, write

$$\omega = F(T)\widehat{\omega}^{\otimes r}$$

where $T = q - 1$ is the canonical uniformizer of $\mathcal{R}$ coming from Serre-Tate theory, and F is a power series in T.

Then it follows from Katz's computation of ∇ and the Leibniz rule that

$$\nabla\omega = \sum_{i=0}^{r-1} F(T)\widehat{\omega}^{\otimes i} \otimes \widehat{\eta} \otimes \widehat{\omega}^{\otimes r-1-i} \otimes \operatorname{dlog} q + F'(T)\widehat{\omega}^{\otimes r}$$

By Lemma IV.2, the splitting Ψ_p sends $\widehat{\eta}$ to 0, so

$$\theta\omega = F'(T)\widehat{\omega}^{\otimes r} \otimes \mathrm{KS}^{-1}(dT).$$

Now

$$dT = q \operatorname{dlog} q = (1 + T) \operatorname{dlog} q,$$

so

$$\theta f = (T + 1)F'(T)\widehat{\omega}^{\otimes r+2}$$

by formula (5.1). Thus

$$\theta f = (T + 1)\frac{d}{dT}F(T)\widehat{\omega}^{\otimes r+2}.$$

This result was originally obtained in the Shimura curve case by Mori ([Mo]).

5.3 Hecke operators in coordinates

We handle T_ℓ for $\ell \neq p$ first. Suppose $\psi : A \to A/C$ is an isogeny of degree prime to p. Then the subgroup C and the map ψ deform uniquely to a subgroup scheme

$\mathcal{C}$ and a map $\psi : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ for any lift $\mathcal{A}$ of A to L. Recall the fixed generator P_1 of eT_pA. Because ψ and ψ^t both induce isomorphisms of p-adic Tate modules, $Q_1^t := (\psi^t)^{-1}(P_1^t)$ is a generator of $eT_p^t(A/C)$. Write $\widehat{\omega}_1 := e\widehat{\omega}_{P_1^t}$ for the canonical formal one-form on the disk $\widetilde{D} \subset X(L)$ whose points correspond to characteristic 0 abelian surfaces reducing to A/C with level structure $\psi(\mathbf{t})$. By Lemma IV.4, It satisfies

$$\psi^*\widehat{\omega}_{\mathbf{1}} = \widehat{\omega}.$$

Given this choice of Q_1^t, there is a corresponding basis element of $T_p(A/C)$ via the canonical principal polarization of A/C, which is $Q_1 = \frac{1}{\deg\psi}\psi^t(Q_1^t)$. Thus there is a Serre-Tate coordinate $\widetilde{q} = q(\_, Q_1, Q_1^t)$ on the disk $\widetilde{D}$.

Lemma V.4. *The function f_ψ on D given by $\mathcal{A} \mapsto \widetilde{q}(\mathcal{A}/\mathcal{C})$ can be computed as*

$$f_\psi = q^{\frac{1}{\deg\psi}}$$

Proof. It suffices to check that the two functions agree for any deformation $\mathcal{A}$ of A to an Artin local ring R with residue field k. It follows from Proposition IV.1 that

$$\begin{aligned} f_\psi(\mathcal{A}) &= q(\mathcal{A}/\mathcal{C}, Q_1, Q_1^t) \\ &= q(\mathcal{A}/\mathcal{C}, \frac{1}{\deg\psi}\psi_t(P_1), Q_1^t) \\ &= q(\mathcal{A}, \frac{1}{\deg\psi}P_1, P_1^t) \\ &= q(\mathcal{A})^{\frac{1}{\deg\psi}} \end{aligned}$$

□

Note that the final expression does not depend on any choices, as the Serre-Tate parameter q is a principal unit and $\deg\psi$ is prime to p.

Corollary V.5. *For each cyclic degree ℓ isogeny $A \to A/C_i$ of A, write $\widehat{\omega}_i$ for the canonical one-form on the disk R_i of points reducing to A/C_i, as normalized above. Suppose f is a modular form such that for each i, the Serre-Tate expansion on the disk D_i is*

$$f = F_i(T_i)\widehat{\omega}_i^{\otimes k}$$

Then on the disk D, $f|T_\ell$ is given by

$$\sum_{i=1}^{\ell+1} F_i((1+T)^{1/\ell} - 1)\widehat{\omega}^{\otimes k}.$$

We move on to the operators U and V. Write ϕ for the mod p Frobenius. For D an ordinary residue disk corresponding to a false elliptic curve A with level structure t, write D^ϕ for the disk corresponding to A^{frob} with level structure $\frac{1}{p}t^\phi$. Note that, because of the extra factor of $\frac{1}{p}$ on the level structure, D^ϕ is *not* the image of D under the map $X^{\text{ord}} \to X^{\text{ord}}$ under the canonical p-isogeny $\mathcal{A} \to \mathcal{A}/\mathcal{C}_0$.

We are going to pick Serre-Tate coordinates on these disks in a compatible way. One cannot ensure, as in the prime-to-p case, that the canonical formal one-form pulls back to the canonical formal one-form. However, using ordinarity, we do at least have that $Q_1 := \phi P_1$ is a basis for eT_pA^ϕ. Write $\{Q_1^t, Q_2^t\}$ for the corresponding basis for $T_pA^{\phi,t}$, using the principal polarization on A. Then one has

$$\phi^t(Q_i^t) = pP_i^t$$

Write q_1 for the function on D^ϕ corresponding to this basis. There is a function f on $D = D_0$ given by $(\mathcal{A}, t) \mapsto q_1(\mathcal{A}/C_{\mathcal{A}}, \frac{1}{p}t^\phi)$, where $\mathcal{C}_{\mathcal{A}}$ is the canonical subgroup of $\mathcal{A}$.

Lemma V.6. *One has*

$$f = q^p$$

as functions on D.

Proof. Again it suffices to check that the two functions agree for any deformation $\mathcal{A}$ of A to an Artin local ring R with residue field k. One has

$$\begin{aligned} f(\mathcal{A}) &= q(\mathcal{A}/C, \phi P_1, Q_1^t) \\ &= q(\mathcal{A}, P_1, \phi^t Q_1^t) \\ &= q(\mathcal{A}, P_1, pP_1^t) \\ &= q(\mathcal{A})^p \end{aligned}$$

□

Let $\widehat{\omega}_0$ be the canonical formal relative one-form for D_0 attached to the Tate-module generators $\{\phi P_1, \phi P_2\}$. Write $\Phi : D \to D_0$ for the "quotient by the canonical subgroup" map.

Lemma V.7. *One has*

$$\Phi^* \widehat{\omega}_0 = p\widehat{\omega}$$

Proof. This follows from Lemma IV.4. □

Corollary V.8. *If $f = \sum F(T)\widehat{\omega}_0^{\otimes k}$ is a modular form on D^ϕ expressed in Serre-Tate coordinates, then the corresponding modular form $f|_V$ is given in Serre-Tate coordinates on D by*

$$f|_V = F((1+T)^p - 1)\widehat{\omega}^{\otimes k}$$

Finally, we compute U. For $1 \leq i \leq p$, write D_i for the image of D under the map $X^{\mathrm{ord}} \to X^{\mathrm{ord}}$ under the (not-canonical) p-isogenies $\mathcal{A} \to \mathcal{A}/\mathcal{C}_i$. Then the map $D \to D_i \to (D_i)^\phi$ is the identity map (because of the factor of $1/p$ in the level structure). For each subgroup $\mathcal{C}$ other than the canonical subgroup of the universal false elliptic curve over D, there is a function g_C given by $(\mathcal{A}, t) \mapsto q_i(\mathcal{A}/C, \mathrm{Image} t)$, where $C = \mathcal{C}_{\mathcal{A}}$.

Lemma V.9. *One has*

$$g_C^p = q$$

Proof. This follows from Lemma V.6, because $(\mathcal{A}/C)^t \to \mathcal{A}^t$ is the canonical isogeny for $(\mathcal{A}/C)^t$. $\square$

Fix a primitive pth root of unity $\zeta \in \mathbb{C}_p$.

Lemma V.10. *There is an ordering on the disks D_i such that on C_i one has*

$$g_{C_i} = \zeta^i q.$$

Proof. It follows from the preceding lemma that there is some root of unity making the lemma true; the content of the lemma is that each distinct root of unity appears exactly once. To figure out which root of unity shows up on the disk D_i, we may evaluate at $q_i = 1$, i.e. at the CM point on the disk D_i. This calculation is done by Brakočević in the GL_2 case, using Shimura's reciprocity law for GL_2 (see the proof of Lemma 7.2 of [Br]).

To reduce to the GL_2 case, note that, by the argument on p. 918 of [Pr], there is a false isogeny (defined over a number field in which p is unramified)

$$\lambda : \mathcal{A} \to \mathcal{E}_1 \times \mathcal{E}_2$$

of $\mathcal{A}$ with a product of elliptic curves, with degree prime to p. The result then follows from Proposition IV.1. $\square$

The pth roots of q in the ring of $\mathbb{C}_p$-valued functions on D are given by Taylor-expanding $\zeta^i(1+T)^{1/p}$. Write $\Phi_i : D \to D_i$ for the map killing the ith (non-canonical) subgroup.

Lemma V.11. *One has* $\Phi^*\widehat{\omega}_i = \widehat{\omega}$

Proof. This follows from Lemma V.7, since the degree of Φ_i is p. $\square$

Proposition V.12. *If* $f = \sum F_i(T)\widehat{\omega}^{\otimes k}$ *is a rigid-analytic modular form on* D_i *expressed in Serre-Tate coordinates, then the corresponding modular form* $f|_U$ *is given in Serre-Tate coordinates on* D *by*

$$f|_U(T) = \frac{1}{p}\sum_{i=0}^{p-1} F(\zeta^i(1+T)^{1/p} - 1)\widehat{\omega}^{\otimes k}.$$

Proof. Just a restatement of the preceding two lemmas. Note that it makes sense to evaluate F at $(\zeta^i(1+T)^{1/p} - 1)$, as the constant coefficient of $\zeta^i(1+T)^{1/p} - 1$ is $\zeta^i - 1$, which lives in the maximal ideal of $W[\zeta]$. $\square$

The importance of the above formulas is that they give a formula for the composition UV of Hecke operators (the composition VU is the identity).

Proposition V.13. *Suppose that* f *has Serre-Tate expansion*

$$f = F(T)\widehat{\omega}^{\otimes k}$$

on the disk D*. Then* $f|_{UV}$ *has Serre-Tate expansion*

$$f|_{UV}(T) = \frac{1}{p}\sum_{i=0}^{p-1} F(\zeta^i(1+T) - 1)\widehat{\omega}^{\otimes k}.$$

Moreover, if $F(T) \in W[[T]]$*, then* $\frac{1}{p}\sum_{i=0}^{p-1} F(\zeta^i(1+T) - 1) \in W[[T]]$.

Proof. Write $f = F_\phi(T)\widehat{\omega}^{\otimes k}$ in Serre-Tate coordinates on the disk D^ϕ.

We compute

$$\begin{aligned}
(F(T)\widehat{\omega}^{\otimes k})|_{UV} &= (F_\phi((1+T)^p - 1)\widehat{\omega}_0^{\otimes k})|_U \\
&= \frac{1}{p}\sum_{i=0}^{p-1} F(\zeta^i(1+(1+T)^p - 1)^{1/p} - 1)\widehat{\omega}^{\otimes k} \\
&= F(\zeta^i(1+T) - 1)\widehat{\omega}^{\otimes k}.
\end{aligned}$$

The integrality claim for this expression is well-known (see e.g. p. 16 of [CS]). To prove it, note that

$$\sum_{i=0}^{p-1} F(\zeta^i(1+T)-1)$$

has coefficients in the maximal ideal $\mathfrak{p}$ of $W[\zeta]$, because it reduces to 0 mod $(1-\zeta)$. Thus the coefficients lie in $\mathfrak{p} \cap W = (p)$. □

5.4 Continuity properties of the operators

Write Θ for the operator $(1+T)\frac{d}{dT}$ on the ring $W[[T]]$, so what we have seen so far is that

$$\theta(F(T)\widehat{\omega}^{\otimes k}) = (\Theta F)(T)\widehat{\omega}^{\otimes k+2}.$$

In this section, we investigate elementary continuity properties of the operator Θ use them to deduce similar properties for θ on the space of p-adic modular forms.

Proposition V.14. *The Θ operator satisfies the continuity condition*

$$\Theta^i F \equiv \Theta^j F \mod p^n$$

for any F and any $i, j \geq k$ such that $i \equiv j \mod (p-1)p^{n-1}$.

Proof. First suppose that F is a polynomial. Setting $x = 1+T$, on the ring $W[x] = W[T] \subseteq W[[T]]$, we have $\Theta = x\frac{d}{dx}$, so $\theta^i \sum a_n x^n = \sum n^i a_n x^n$ and the result follows (using Fermat's little theorem for the terms with $(p,n) = 1$ and the condition $i, j \geq k$ for the others).

To establish the result for a general power series $F = \sum b_n T^n$, we may fix n and prove that the congruence holds for the coefficients of T^n in $\Theta^i F$ and $\Theta^j F$. Note that the coefficient of T^n in ΘF depends only on the coefficients of T^n and T^{n+1} in F. Thus, the coefficient of T^n in $\Theta^i F$ depends only on the numbers $b_n, b_{n+1}, \ldots, b_{n+i}$, and similarly for $\Theta^j F$. It follows that there exists a polynomial truncation G of F

such that the coefficients of T^n in $\Theta^i G$ and $\Theta^j G$ are the same as those for F. Since the congruence holds for polynomials, the result follows.

□

Corollary V.15. *Suppose that f is a p-integral modular form, i.e., that f is a modular form over some subring of W. Then for any ordinary pair (A, ω) one has*

$$\theta^i f(A, \omega) \equiv \theta^j f(A, \omega) \mod p^n$$

whenever $i \equiv j \mod (p-1)p^{n-1}$.

Write $f^\flat = f|_{VU-UV}$, and similarly for $F \in W[[T]]$ we write $F^\flat = F|_{\mathbf{VU}-\mathbf{UV}}$, where **UV** is the formal operator on power series of Theorem V.13 and **VU** is the identity operator.

Proposition V.16. *One has*

$$F^\flat = \lim_{i\to\infty} \Theta^{p^i(p-1)} F.$$

Proof. The limit on the right hand side makes sense because of Proposition V.14. Writing $\Theta^{(p-1)p^\infty}$ for the operator $\lim_{i\to\infty} \Theta^{p^i(p-1)}$, we see that $\Theta^{(p-1)p^\infty}$ is a continuous W-linear operator on $W[[T]]$. As this is also the case for the operator $\flat$, it suffices to check the putative equality on the polynomials $F_m = (1+T)^m$, since the linear span of these polynomials is dense. One has $\Theta^i F_m = m^i F_m$, and so

$$\Theta^{(p-1)p^\infty} F_m = \begin{cases} 0, & p|m \\ F_m, & (p,m) = 1. \end{cases}$$

On the other hand, using Proposition V.13, we compute

$$\begin{aligned} F_m^\flat(T) &= F_m(T) - \frac{1}{p}\sum_{i=0}^{p-1} F_m(\zeta^i(1+T)-1) \\ &= F_m(T) - \frac{1}{p}\sum_{i=0}^{p-1} \zeta^{mi}(1+T)^m. \end{aligned}$$

If p is prime to m, then the sum is zero, since ζ^{mi} ranges over a complete set of pth roots of unity. If p divides m, the sum is $F_m(T)$. In either case $F_m^\flat = \Theta^{(p-1)p^\infty} F_m$ as desired. $\square$

We return to the operators θ, U, and V on the space of p-adic modular forms. If f is a p-adic modular form, and (A', t', ω') is a triple consisting of a false elliptic curve over L with ordinary reduction, level structure, and a translation-invariant one-form, then the limit

$$\lim_{i\to\infty} \theta^{p^i(p-1)} f(A', t', \omega')$$

exists and equals $f^\flat(A', t')$, since this statement can be checked on residue disks. In particular, if j is a negative integer, it makes sense to write

$$\theta^j f(A', t', \omega')$$

to mean

$$\lim_{i\to\infty} \theta^{j+p^i(p-1)} f.$$

A priori, $\theta^j f$ is a locally analytic modular form, rigid when restricted to a fixed residue disk. Note that, in spite of the notation, one has $\theta^k \theta^{-k} f = f^\flat$, not f.

5.5 Proof of Proposition III.6

We conclude by proving the formula

$$(\theta f)|_{T_\ell} = \ell\theta(f|_{T_\ell})$$

as promised. In each case the result follows from the explicit formulas for the Hecke operators (using the chain rule). For $\ell \neq p$, this is a simple calculation. For T_p it will follow from related formulas for U and V, using the formula

$$T_p = U + \frac{1}{p}[p]V.$$

Letting ψ denote the automorphism of X mapping (A, t) to (A, pt), it follows directly from the modularity of f that

$$f|_{[p]} = p^k f \circ \psi,$$

and thus

$$\theta(f|_{[p]}) = p^{-2}(\theta f)|_{[p]}$$

(because θ boosts the weight of f by 2). The chain rule argument gives that

$$\theta(f|_V) = p(\theta f)|_V$$

and

$$p\theta(f|_U) = (\theta f)|_U,$$

which is the desired result.

CHAPTER VI

Twisted cohomology groups and residue theory

6.1 Review of Deligne's twisted cohomology groups

Let $X/\mathbb{C}$ be a variety, and suppose that $\mathbb{V}$ is a local system on $X(\mathbb{C})$, that is, a sheaf locally (for the complex topology) isomorphic to the constant sheaf $\mathbb{C}^g$. Deligne [De] then showed how to recover the cohomology groups $H^i(X(\mathbb{C}), \mathbb{V})$ algebraically, generalizing the case $\mathbb{V} = \mathbb{C}$ of algebraic de Rham cohomology. Recall that the vector bundle $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_X$ is algebraic, as is the connection $\mathcal{V} \to \mathcal{V} \otimes \Omega$ for which $\mathbb{V}$ is the sheaf of horizontal sections (see Section 3.2.1). Then $H^i(X(\mathbb{C}), \mathbb{V})$ coincides with the hypercohomology of the complex

$$0 \xrightarrow{\nabla} \mathcal{V} \otimes \Omega \xrightarrow{\nabla} \mathcal{V} \otimes \wedge^2 \Omega \xrightarrow{\nabla} \dots \tag{6.1}$$

Write $H^i_{\mathrm{dR}}(X, \mathcal{V}, \nabla)$ for the ith hypercohomology group of this complex. Of course these algebraic definitions all make sense over an arbitrary base field k (they are not useful unless the characteristic of k is zero).

Given two vector bundles $\mathcal{V}, \mathcal{V}'$ with flat connections ∇, ∇', the natural map of the complexes (6.1) for $\mathcal{V}, \mathcal{V}'$, and $\mathcal{V} \otimes \mathcal{V}'$ gives rise to maps

$$H^i(X, \mathcal{V}, \nabla) \otimes H^j(X, \mathcal{V}', \nabla') \to H^{i+j}(X, \mathcal{V}, \nabla \otimes \nabla').$$

We will apply this observation in particular to the case where $(\mathcal{V}, \nabla)$ is self-dual, so that the target is the ordinary algebraic de Rham cohomology of X.

As in the case where $\mathbb{V}$ is the trivial local system, we get a "Hodge sequence"

$$0 \to H^0(X, \mathcal{V} \otimes \Omega) \to H^1(X, \mathcal{V}, \nabla) \to H^1(X, \mathcal{V}),$$

as can easily be seen by taking a Cech resolution of (6.1).

The remaining sections of this chapter are used only in the proof of Proposition VIII.1, which is extremely similar to the proof of Proposition 3.18 of [BDP]. The reader who is already familiar with the arguments in [BDP], or who wishes to take the statement of Proposition VIII.1 on faith, may prefer to skip to the next chapter.

6.2 Residues, algebraically

If X is a smooth curve over a field k, Tate gave an algebraic definition of the residue of a meromorphic one-form at a point. Residue theory gives us a canonical way to compute the algebraic de Rham cohomology of X. As is well-known, these results carry over to meromorphic forms valued in vector bundles on X with flat connections.

Throughout this section, $\mathcal{V}$ is a vector bundle on X with a connection ∇ which is *integrable* (flat). For P a point of X, write $\mathcal{V}(P)$ for the fiber of $\mathcal{V}$ above P.

Definition VI.1. If $U \subseteq X$ is open, then ∇ *trivializes* on U if $\mathcal{V}(U)$ admits a basis of horizontal sections for ∇.

Remark VI.2. The condition that ∇ trivializes on U is stronger than the condition that $\mathcal{V}$ trivializes on U. For example, a connection on the trivial bundle $\mathcal{O}$ is none other than a one-form ω, and the connection corresponding to ω trivializes on an open set U if and only if there is a function f on U solving the differential equation

$$\text{dlog} f = \omega. \tag{6.2}$$

If a connection admits a trivialization on an open set U, we may define the *residue* of a meromorphic section at a point $P \in U$ by stringing together the maps

$$\mathcal{V} \otimes \Omega \otimes K(U) \xrightarrow{f} \widehat{\Omega}^n(U) \xrightarrow{\mathrm{res}_P} k^n = (\mathcal{O}^n)(U)^{d=0} \xrightarrow{f^{-1}} \mathcal{V}(U)^{\nabla=0} = \mathcal{V}(P), \tag{6.3}$$

where res_P denotes the Tate residue map (formally expand in a parameter t at P and take the coefficient of t^{-1}).

The residue does not depend on the choice of trivialization because any choice differs by an element of $\mathrm{GL}_n(\mathbb{C})$, for which the classical residue map is linear (note that if we insisted only on trivializing the vector bundle and not the connection, the ambiguity would be up to $\mathrm{GL}_n(\mathcal{O})$, and the residue would depend on the choice).

As shown already, the Gauss-Manin connection admits local trivializations in the rigid topology, but not in the Zariski topology. However, in characteristic zero, *any* integrable connection admits a trivialization in a formal neighborhood of a point: that is, writing $\widehat{\mathcal{O}_P}$ for the completion of the local ring at P and $\widehat{\mathcal{V}_P}$ for the completion of the stalk of $\mathcal{V}$ at P, there exists a trivialization f making the following square commute:

$$\begin{tikzcd} \widehat{\mathcal{V}_P} \arrow[r, "f"] \arrow[d, "\nabla"] & \widehat{\mathcal{O}_P^n} \arrow[d, "d"] \\ \widehat{\mathcal{V} \otimes \Omega_P} \arrow[r, "f"] & \widehat{\Omega_P^n} \end{tikzcd} \tag{6.4}$$

To see this, pick a trivialization of $\mathcal{V}$ in a formal neighborhood of a point, and then write down a formal solution to Equation 6.2. Since this trivialization always exists, we use it, rather than (6.3) in the definition below of residues for arbitrary vector bundles with flat connection; in practice, the residues in this document are computed in the rigid topology.

Definition VI.3. The *residue* of a meromorphic section of $\mathcal{V} \otimes \Omega$ with connection

∇ at a point P is the element of the fiber of $\mathcal{V}$ at P given by the maps

$$\mathcal{V}\otimes\widehat{\Omega\otimes K}_P \xrightarrow{f} \widehat{\Omega}^n_P \xrightarrow{\mathrm{res}_P} k^n = (\widehat{\mathcal{O}}^n_P)^{d=0} \xrightarrow{f^{-1}} \widehat{\mathcal{V}_P}^{\nabla=0} = \mathcal{V}(P)$$

where f is a choice of trivialization.

Proposition VI.4. *The twisted first algebraic de Rham cohomology*

$$H^1(X,\mathcal{V},\nabla)$$

can be computed as the space of ∇-closed $\mathcal{V}$-valued one-forms with residue zero at every pole modulo ∇-exact forms.

Proof. This is a direct generalization of Tate's technique for computing the algebraic de Rham cohomology of curves, which is the case $\mathcal{V}=\mathcal{O}$ and $\nabla=d$. Write K for the function field of X, thought of as a constant sheaf on X. Then the complex

$$0\to\mathcal{V}\xrightarrow{\nabla}\mathcal{V}\otimes\Omega\to 0$$

has a resolution coming from the double complex

$$\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d] & \mathcal{V}\otimes\Omega \arrow[d] \\
\mathcal{V}\otimes K \arrow[r] \arrow[d] & \mathcal{V}\otimes\Omega_{K/k} \arrow[d] \\
\bigoplus\limits_{P\in U}\dfrac{\widehat{(\mathcal{V}\otimes K)}_P}{\mathcal{V}_P} \arrow[r] & \bigoplus\mathcal{V}_P\otimes\Omega_{\frac{K_P}{\mathcal{O}_P}}
\end{tikzcd}$$

where all horizontal arrows are induced by ∇. The condition that a meromorphic $\mathcal{V}$-valued form have residue zero at every pole is exactly the condition that it have ∇-primitives in the completion of $\mathcal{V}$ at every point. Such primitives are well defined up to ∇-horizontal, in particular holomorphic, sections, so the result follows as in the classical case. $\square$

Finally, suppose that $\mathcal{V}$ is self-dual. Then we get a cup product pairing on hypercohomology:

$$H^1(X, \mathcal{V}, \nabla) \otimes H^1(X, \mathcal{V}, \nabla) \to H^2_{\mathrm{dR}}(X) = k,$$

which can be computed by the rule

$$(\lambda, \mu) = \sum \mathrm{res}_P \langle L_P, \mu_P \rangle.$$

where L_P denotes a ∇-primitive for λ_P, defined in some neighborhood of P, and the quantity on the right is the residue of an honest meromorphic one-form.

6.3 Coleman's rigid analytic theory of residues

Let $X/\mathbb{C}_p$ be a curve with good reduction and $\mathcal{V}$ a vector bundle on X with flat connection. Write X^{rig} for the associated rigid analytic space and $\mathcal{V}^{\mathrm{rig}}$ for the analytification of $\mathcal{V}$.

For any point $P \in X(\mathbb{C}_p)$, write D_P for the residue disk containing P, which is isomorphic as a rigid space to the open unit disk. Fixing one such isomorphism taking P to 0 allows us to speak of the affinoid subdomain of D_P given by "the closed disk of radius r" for any $r < 1$, denoted $D_{P,r}$. Fixing a finite number of points $P_1, \ldots, P_n$, consider the affinoid space

$$\mathcal{A} = X^{\mathrm{rig}} \setminus D_{P_1} \setminus \ldots \setminus D_{P_n}$$

and its "basic wide-open neighborhoods" (for various choices of radii r_i with $0 < r_i < 1$)

$$\mathcal{W}_{r_1,\ldots,r_n} = X^{\mathrm{rig}} \setminus D_{P_1,r_1} \setminus \ldots \setminus D_{P_n,r_n}.$$

For ω a one-form on an annulus, Coleman has defined a notion of "residue" which coincides with the algebraic notion of residue when ω comes from an algebraic one-form. Using this definition, together with the formalism of the previous section, it similarly makes sense to speak of residues of vector-bundle-valued one-forms on annuli, provided that the vector bundle comes with a flat connection that trivializes on sufficiently small disks. Coleman's residue is only well-defined up to a sign (depending on the "orientation" of the annulus), but he shows in Corollary 3.7a of [Co2] that one can compatibly orient all the annuli $D_{P_i} \setminus D_{P_i,r_i}$ by choosing a uniformizer of the deleted point as a uniformizer in the ring of rigid functions on the annulus (instead of choosing the reciprocal of a uniformizer). Here "compatibility" implies that the residue of a meromorphic one-form on a Zariski open will agree with the residue of the same form thought of as a rigid one-form on an annulus, rather than with its negative.

Coleman has shown that the algebraic de Rham cohomology of the affine curve $X \setminus \{P_1, \ldots, P_n\}$ can be computed analytically as the "honest" de Rham cohomology of any wide open neighborhood of $\mathcal{A}$:

Theorem VI.5 (Coleman)**.** *For any wide open neighborhood $\mathcal{W}$, the natural map*

$$H^1_{dR}(X \setminus \{P_1, \ldots, P_n\}, \mathcal{V}, \nabla) \to \frac{\mathcal{V}_{\mathcal{W}} \otimes \Omega_{\mathcal{W}}}{\nabla \mathcal{V}_{\mathcal{W}}}$$

is an isomorphism.

Write $H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{V}, \nabla)$ for

$$\frac{\mathcal{V}_{\mathcal{W}} \otimes \Omega_{\mathcal{W}}}{\nabla \mathcal{V}_{\mathcal{W}}}.$$

The following two immediate consequences of Coleman's theory are immediate:

Corollary VI.6. *Any inclusion of wide-opens $\mathcal{A} \subset \mathcal{W} \subset \mathcal{W}'$ induces an isomorphism*

$$H^1_{dR}(\mathcal{W}', \mathcal{V}, \nabla) \xrightarrow{\sim} H^1_{dR}(\mathcal{W}, \mathcal{V}, \nabla).$$

Corollary VI.7. *The image of the natural map*

$$H^1_{dR}(X, \mathcal{V}, \nabla) \to \frac{\mathcal{V}_{\mathcal{W}} \otimes \Omega_{\mathcal{W}}}{\nabla \mathcal{V}_{\mathcal{W}}}$$

is the space of classes of rigid 1-forms on $\mathcal{W}$ with residue zero at each of the points P_i.

If $\mathcal{V}$ is a vector bundle with flat connection, then a *primitive* for a $\mathcal{V}$-valued one-form ω (over some open set) is a section F_ω of $\mathcal{V}$ with $\nabla F_\omega = \omega$. Of course, a primitive is only unique up to horizontal sections of $\mathcal{V}$. In the p-adic setting, Coleman has shown a canonical way to write down a primitive for sections of $\mathcal{V}$ in the event that $\mathcal{V}$ is equipped with some extra structure coming from the Frobenius map on the reduction of X. For more details on the following, the reader should consult Section 10 of [Co].[1]

The reduction X_p^0 of $\mathcal{A}$ is a smooth affine curve which admits the p-power absolute Frobenius map to itself. If the set $\{P_1, \ldots, P_n\}$ is Frobenius stable (which one may always assume by adding more points to it), then, using the good reduction hypotheses, Coleman shows this Frobenius map lifts to a semilinear map $\phi : \mathcal{A} \to \mathcal{A}$. Fix a choice of such a ϕ once and for all.

Definition VI.8. A Frobenius neighborhood of $\mathcal{A}$ in $\mathcal{W}$ is a pair $(\mathcal{W}', \phi)$, where $\mathcal{W}'$ is a basic wide open neighborhood with $\mathcal{A} \subset \mathcal{W}' \subset \mathcal{W}$ and $\phi : \mathcal{W}' \to \mathcal{W}$ restricts to ϕ on $\mathcal{A}$.

Definition VI.9. An *overconvergent Frobenius isocrystal* on an affinoid $\mathcal{A}$ is a pair $(\mathcal{V}, \mathrm{Fr})$ where $\mathcal{V}$ is a vector bundle with flat connection ∇ on $\mathcal{W}$ and Fr is a ∇-horizontal morphism

$$\mathrm{Fr} : \phi^*\mathcal{V}|_{\mathcal{W}'} \to \mathcal{V}|_{\mathcal{W}'}$$

on some Frobenius neighborhood $\mathcal{W}'$ of $\mathcal{A}$ in $\mathcal{W}$.

[1] In that section, Coleman uses the phrase "over-convergent F-crystal" to mean what this document and others call an "overconvergent Frobenius isocrystal." Moreover, Coleman does not limit his theory to the good-reduction case, which requires him to distinguish between the "flab"-sheaf of locally analytic sections of $\mathcal{V}$ and a certain "flog" sub-sheaf.

Given an overconvergent Frobenius isocrystal $\mathcal{V}$, there is an endomorphism Φ of the space

$$H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{V}, \nabla)$$

given by the composition

$$H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{V}, \nabla) \to H^1_{\mathrm{dR}}(\mathcal{W}', \phi^*\mathcal{V}, \nabla) \to H^1_{\mathrm{dR}}(\mathcal{W}', \mathcal{V}, \nabla) = H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{V}, \nabla).$$

Definition VI.10. A polynomial $P(T) \in F[T]$ is a *Coleman polynomial* for a class $[\omega] \in H^1(\mathcal{W}, \mathcal{V}, \nabla)$ if the following hold:

- $P(\Phi^*)([\omega]) = 0$.
- $P(\Phi)$ induces an automorphism of the space of locally analytic sections of $\mathcal{V}$ that are horizontal for ∇.
- $P(1) \neq 0$.

Theorem VI.11 (Coleman)**.** *Suppose that ω is a $\mathcal{V}$-valued one-form on $\mathcal{W}$ such that the cohomology class $[\omega] \in H^1(\mathcal{W}, \mathcal{V}, \nabla)$ admits a Coleman polynomial $P(T)$. Then there is a unique locally analytic primitive F_ω for ω such that $P(\Phi^*)F_\omega$ is a rigid section of $\mathcal{V}$ on some Frobenius neighborhood of $\mathcal{A}$ in $\mathcal{W}$. Moreover, F_ω is rigid on any fixed residue disk of $\mathcal{A}$.*

The function F_ω is called the *Coleman primitive* for ω. It turns out that it depends on none of the choices involved in stating Theorem VI.11 – that is, it does not depend on $\mathcal{W}'$, the extension of ϕ to $\mathcal{W}'$, nor on the Coleman polynomial (provided that a polynomial exists). Of course, we will apply this theory in the setting of $\mathcal{L}_{2r}$ on the ordinary locus of C (and its trivial enlargement $\mathcal{L}_{2r,2r}$). In this case, the overconvergent Frobenius isocrystal structure comes from an extension of the canonical morphism $\mathcal{A}_{\mathrm{ord}} \to \mathcal{A}_{\mathrm{ord}}$ to a wide open neighborhood $\mathcal{W}$ of the ordinary locus, as is constructed in Chapter 11 of [Kas]. It follows as on p. 42 of [Co] that there is a Coleman polynomial for the $\mathcal{L}_{2r}$-valued differential form which is the class of any newform of weight $2r+2$.

The following lemma will be useful in the proof of Proposition VIII.1.

Lemma VI.12. *Given an overconvergent Frobenius isocrystal* $(\mathcal{V}, \mathrm{Fr})$ *on an affinoid* $\mathcal{A}$*, a pairing on* $\mathcal{V}$ *that is compatible with the connection,* $[\omega] \in H^1(\mathcal{W}, \mathcal{V}, \nabla)$ *a cohomology class on a wide open neighborhood of* $\mathcal{A}$ *admitting a Coleman primitive* F_ω*, and* $[\eta^{frob}] \in H^1(\mathcal{W}, \mathcal{V}, \nabla)$ *a cohomology class that is fixed by Frobenius, one has*

$$\sum_{i=1}^{n} \mathrm{res}_{P_i} \langle F_\omega, \eta^{frob} \rangle = 0.$$

Proof. This is Lemma 3.20 of [BDP]. □

CHAPTER VII

Construction of the cycle

7.1 Projectors on Kuga-Sato varieties and the cohomology of Shimura curves

Recall that C is the Shimura curve over F, f is a modular form of weight $k = 2r+2$ on C, $\mathcal{A}_r$ is the r-fold fiber product of the universal false elliptic curve over C with itself, A is a fixed "CM false elliptic curve," and $W_r = \mathcal{A}_r \times A^r$. In this chapter we construct a homologically trivial cycle on W_r, then begin our discussion of the p-adic Abel-Jacobi map, as applied to this cycle. For the first two sections we assume $r > 0$. The case $r = 0$ (so $W_r = C$) is treated separately in a section at the end of this chapter.

As in [BDP], our cycle will be the graph of a morphism of false elliptic curves, modified by an idempotent in the ring of correspondences on a Kuga-Sato variety. All rings of correspondences in this chapter are taken with *rational* coefficients.

Recall the bundle $\mathcal{L}_{2r} = \mathrm{Sym}^{2r}\, e\mathcal{H}^1$.

Theorem VII.1 (Besser)**.** *There is a projector P in the ring*

$$\mathrm{Corr}_C(\mathcal{A}_r)$$

of algebraic correspondences on $\mathcal{A}_r$ fibered over C, with the property

$$P\mathcal{H}^*(\mathcal{A}_r/C) = P\mathcal{H}^{2r}(\mathcal{A}_r/C) = \mathcal{L}_{2r}.$$

Proof. For completeness, we outline Besser's construction here. The construction is a bit lengthy, and is broken into a series of small propositions. None of the properties of P except the one quoted in the statement of the theorem will be used in the sequel. Therefore, the reader approaching the subject for the first time may wish to grant the statement of this theorem and skip to Lemma VII.4. Besser makes the projector in the following steps:

1. There is a projector Q in the ring of C-correspondences on $\mathcal{A}$ with $Q\mathcal{H}^2(\mathcal{A}/C) = \operatorname{Sym}^2 e\mathcal{H}^1(\mathcal{A}/C)$.

2. There is a Künneth projector P_1 in the ring of C-correspondences on $\mathcal{A}_r$ with $P_1\mathcal{H}^*(\mathcal{A}_r/C) = \operatorname{Sym}^r Q\mathcal{H}^2(\mathcal{A}/C)$.

3. There is a projector P_2 in the ring of C-correspondences on $\mathcal{A}_r$ with $P_2 \operatorname{Sym}^r Q\mathcal{H}^2 = \operatorname{Sym}^{2r} e\mathcal{H}^1(\mathcal{A}/C)$ (under the identification from Step 1).

For step one, a calculation (which may be performed fiber for fiber and thus after choosing a trivialization of $B \otimes K$ and a basis) identifies $\operatorname{Sym}^2 e\mathcal{H}^1(\mathcal{A}/C)$ with the common kernel of the operators $b - \mathbf{N}(b)$ acting on $\mathcal{H}^2(\mathcal{A}/C)$.

The existence of the Künneth projector (step two) is a famous theorem of Deninger and Murre, which holds for an arbitrary abelian scheme (more precisely, one uses the work of Deninger and Murre to project to $\otimes^r Q\mathcal{H}^1$, then applies a standard symmetrizing projector).

In explaining step three, we will not follow the notation of the document at large. Fix a field L of characteristic zero and let V be a two-dimensional vector space over L. By convention, $\operatorname{Sym}^k V$ denotes the *subspace* of $\bigotimes^k V$ fixed by the action of S_k. Let r be a positive integer. There is a natural inclusion

$$\operatorname{Sym}^{2r} V \to \operatorname{Sym}^r \operatorname{Sym}^2 V$$

where $\bigotimes^r \bigotimes^2 V$ is identified with $\bigotimes^{2r} V$ without reordering factors.

Now fix a non-degenerate alternating form $\langle\ ,\ \rangle$ on V. This gives a map $\bigotimes^r \bigotimes^2 V \to \bigotimes^{r-2} \bigotimes^2 V$ by the rule

$$(v_{11} \otimes v_{12}) \otimes \ldots \otimes (v_{r1} \otimes v_{r2}) \mapsto$$
$$\sum_{i<j} \langle v_{i1}, v_{j1}\rangle \langle v_{i2}, v_{j2}\rangle \Big((v_{11} \otimes v_{12}) \otimes \ldots \otimes (\widehat{v_{i1} \otimes v_{j1}}) \otimes \ldots \otimes (\widehat{v_{i2} \otimes v_{j2}}) \otimes \ldots \otimes (v_{r1} \otimes v_{r2}) \Big).$$

The restriction of this map to $\operatorname{Sym}^r \operatorname{Sym}^2 V$ lands in $\operatorname{Sym}^{r-2} \operatorname{Sym}^2 V$; call this restricted map Δ.

Proposition VII.2. *The natural inclusion* $\operatorname{Sym}^{2r} V \to \operatorname{Sym}^r \operatorname{Sym}^2 V$ *identifies* $\operatorname{Sym}^{2r} V$ *with the kernel of* Δ.

Proof. By dimension-counting, it suffices to show that Sym^{2r} is in the kernel of Δ and Δ is surjective. For the former claim, write

$$w = \frac{1}{(2r!)} \sum_{\sigma \in S^{2r}} \sigma w$$

and note that, for fixed $i < j$, the contribution from σw to Δw cancels the contribution from $\sigma \tau w$, where τ is the transposition interchanging the positions i_1 and j_1.

For the surjectivity, write W for $\operatorname{Sym}^2 V$. The form $\langle\ ,\ \rangle$ induces a *symmetric* form on W; we may assume $L = \mathbb{C}$ and pick an orthonormal basis x, y, z for W. With respect to this basis, the operator Δ is just (a scalar multiple of) the usual Laplace operator $\sum \frac{\partial^2}{\partial x_i^2}$, for which the result is well-known. $\square$

Now suppose that V and W are finite dimensional vector spaces equipped with non-degenerate bilinear forms $\langle\ ,\ \rangle$. If $T : V \to W$ is surjective, then

$$\operatorname{Ker}(T^* T) = \operatorname{Ker} T$$

and

$$\operatorname{Im}(T^*T) = (\operatorname{Ker} T)^{\perp}.$$

(The first claim is clear, and anything of the form T^*Tv is certainly orthogonal to the kernel of T. The second claim then follows by dimension counting.)

Lemma VII.3. *Let $f : V \to V$ be an endomorphism. Suppose $f|_{\operatorname{Im} f}$ is injective, and write P for the characteristic polynomial of $f|_{\operatorname{Im} f}$. Then*

$$-\frac{P(f)}{\det f|_{\operatorname{Im} f}}$$

is a projector onto the kernel of f.

Proof. The Cayley-Hamilton theorem implies that $P(f)$ maps V to $\ker f$, and $\frac{-P(f)}{\det f}$ is the identity on $\ker f$ because $P(f)|_{\ker f} = P(0) = -\det f|_{\ker f}$. □

In particular, let P be the characteristic polynomial of $\Delta^*\Delta|_{(\operatorname{Ker}\Delta)^{\perp}}$. Then $P(\Delta^*\Delta)$ is a projector onto $\operatorname{Sym}^{2r} V$.

By the preceding (applied fiber-for-fiber), Step Three of Besser's construction then follows from the fact that $\Delta : \mathcal{H}^{2r}(\mathcal{A}_r/C) \to \mathcal{H}^{2r}(\mathcal{A}_{r-2}/C)$ is induced by a C-morphism $\mathcal{A}_{r-2} \to \mathcal{A}_r$. To see this last fact, consider the morphisms

$$\phi_{ij} : \mathcal{A}_{r-2} \times_C \mathcal{A} \to \mathcal{A}_r$$

given in coordinates by

$$(a_1, \ldots, a_{r-2}) \times a \mapsto (a_1, \ldots, a_{i-1}, a_i, a_i, \ldots, a_{j-1}, a, a_j, \ldots)$$

Then

$$\Delta = \sum_{i,j} \phi_{ij}^*$$

□

Lemma VII.4. *For any $r > 0$,*

$$H^0(C, \mathcal{L}_{2r}, \nabla) = 0.$$

Proof. This can be computed this after base changing to $\mathbb{C}$, and thus (thanks to GAGA for differential operators as in [De]) it suffices to show it for the local system

$$\mathrm{Sym}^{2r} eR^1\pi_*\mathbb{Q}$$

on the Riemann surface $C(\mathbb{C})$. This local system corresponds to the representation $\mathrm{Sym}^{2r} e(\mathbb{C}^4)$ of $\pi_1(C_{\mathbb{C}}) = \Gamma$ for which there are no fixed points. □

Corollary VII.5. *The projector P satisfies*

$$PH^*_{dR}(\mathcal{A}) = H^1(C, \mathcal{L}_{2r}, \nabla)$$

Proof. We first show

$$PH^{2r+1}_{\mathrm{dR}}(\mathcal{A}) = \bigoplus_{p+q=2r+1} H^p(C, P\mathcal{H}^q, \nabla) = H^1(C, \mathcal{L}_{2r}, \nabla).$$

The first equality, known (without the P) as Lieberman's trick, is true for any abelian scheme $X \to S$. Lieberman's observation is that the Leray spectral sequence

$$E_2^{p,q} = H^p(S, \mathcal{H}^q(X/S), \nabla) \Rightarrow H^{p+q}_{\mathrm{dR}}(X)$$

degenerates at page 2, as the multiplication maps $[m] : X \to X$ must on the one hand commute with the edge maps but on the other induce multiplication by m^q on $E_2^{p,q}$. This identifies $H^p(S, \mathcal{H}^q(X/S), \nabla)$ with the m^q eigenspace of $[m]$ on $H^{p+q}_{\mathrm{dR}}(X)$. The second equality follows from Proposition VII.1.

To see that

$$PH^*_{\mathrm{dR}}(\mathcal{A}) \subset H^{2r+1}_{\mathrm{dR}}(\mathcal{A}),$$

observe that P annihilates $H^p_{\mathrm{dR}}(C, \mathcal{H}^q, \nabla)$ unless $p = 0, 1$ and $q = 2r$. As $r > 0$ the latter bundle has no global sections by Lemma VII.4. □

There is a natural inclusion

$$H^0(X, \underline{\omega}^{\otimes r} \otimes \Omega_C) \to H^1_{\mathrm{dR}}(C, \mathcal{L}_{2r}, \nabla)$$

coming from the hypercohomology spectral sequence. This inclusion identifies the former vector space with $\mathrm{Fil}^1 = \mathrm{Fil}^2 = \ldots = \mathrm{Fil}^{2r}$ of the latter.

Proposition VII.6. *There is a projector $\epsilon_A \in \mathrm{Corr}^{2r}(A^r, A^r)$ such that*

$$\epsilon_A H^*_{dR}(A^r) = \mathrm{Sym}^{2r} eH^1_{dR}(A)$$

Proof. Take the image of Besser's projector P under the "evaluation at τ" map

$$\mathrm{Corr}_C(W_r) \to \mathrm{Corr}_F(A^r)$$

□

Consider the variety W_r together with the projector $\epsilon = P\epsilon_{A^r}$ in $\mathrm{Corr}^r_X(W_r, W_r)$. Recall the local system $\mathcal{L}_{2r,2r}$ on C which is $\mathcal{L}_{2r} \otimes \mathrm{Sym}^{2r} eH^1_{\mathrm{dR}}(A)$; the fiber of $\mathcal{L}_{2r,2r}$ at a point P of $C(F)$ corresponding to a false elliptic curve A' is $\mathrm{Sym}^{2r} eH^1(A') \otimes \mathrm{Sym}^{2r} eH^1_{\mathrm{dR}}(A)$.

Proposition VII.7. *One has*

$$\epsilon H^*_{dR}(W_r) = H^1(C, \mathcal{L}_{2r}, \nabla) \otimes \mathrm{Sym}^{2r} eH^1_{dR}(A) \subseteq H^{2r+1}(\mathcal{A}^r) \otimes H^{2r}(A^r) \subseteq H^{4r+1}(W_r).$$

Proof. This follows immediately from the Künneth formula. □

7.2 The generalized Heegner cycle and the p-adic Abel-Jacobi map

For any N^+-torsion point P_A on A and any false isogeny $\phi : A \to A'$ whose kernel intersects the subgroup generated by P_A trivially, there is a point $P_{A',(\phi(P_A)}$ on C, and an embedding of the graph Γ_ϕ into the fiber $A \times A'$ of W_1 above this point. Write Υ_ϕ for the rth power of Γ_ϕ. The cycles studied in this paper are given by

$$\Delta_\phi = \epsilon \Upsilon_\phi.$$

Note that Υ_ϕ has codimension $2r+1$ in W_r. It follows that the cycle class map takes Υ_ϕ to $H^{4r+2}_{\mathrm{dR}}(W_r)$, and so Δ_ϕ is cohomologically trivial by Proposition VII.7.

7.2.1 The p-adic Abel-Jacobi map

Write F_p for the completion of F at the place above p induced by the chosen embedding $\overline{Q} \to \mathbb{C}_p$. Recall that for a variety X, $\mathrm{CH}^i_0(X)$ denotes the group of homologically trivial cycles of codimension i, modulo rational equivalence.

There is a sequence for ètale cohomology with supports: for a closed immersion $Z \hookrightarrow X$ of schemes with complement U, then for any sheaf $\mathcal{F}$ one has

$$\ldots \to H^i_Z(X, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \to H^{i+1}_Z(X, \mathcal{F}) \to \ldots$$

If Z and X are smooth over an algebraically closed field and $\mathcal{F}$ is a locally constant ℓ^n-torsion sheaf, then there are also (functorial) Gysin maps computing the relative cohomology groups in terms of the ordinary cohomology groups of Z; writing c for the codimension of Z in X, the Gysin map identifies

$$H^{j-2c}(Z, \mathcal{F}(-c)) = H^j_Z(X, \mathcal{F}).$$

To apply these general facts, write W_P for the fiber of W_r above P (which has codimension 1) and $W^\flat$ for its complement. By construction, the cycle Δ_ϕ is supported on W_P. Choose $i = 4r+1$ and $\mathcal{F} = \mathbb{Z}_p((2r+1))$. After base changing to the algebraic closure and applying ϵ to the Gysin sequence, we get an exact sequence of

Galois modules

(7.1)

$$0 \to \epsilon H^{4r+1}(\overline{W_r}, \mathbb{Q}_p((2r{+}1))) \to \epsilon H^{4r+1}(\overline{W^\flat}, \mathbb{Q}_p((2r{+}1)) \to \epsilon H^{4r}(\overline{W_P}, \mathbb{Q}_p((2r{+}1){-}1) \to 0$$

using Proposition VII.7 for exactness at the left and right.

There is a map $\mathbb{Q}_p \to \epsilon H^{4r}(\overline{W_P}, \mathbb{Q}_p((2r{+}1){-}1)$ sending 1 to the class of Δ_ϕ. Define $\xi \in \mathrm{Ext}^1_{\text{Galois modules}}(\mathbb{Q}_p, \epsilon H^{4r+1}(\overline{W_r}, \mathbb{Q}_p((2r+1))) = H^1(F_p, \epsilon H^{4r+1}(\overline{W_r}, \mathbb{Q}_p((2r+1)))$ by pushing out the sequence (7.1) along this map.

Write V for the Galois representation $\epsilon H^{4r+1}(\overline{W_r}, \mathbb{Q}_p((2r+1))$. It follows from work of Nekovar ([Ne1]) that this class ξ lies in the subgroup $H^1_f(K, \epsilon H^{4r+1}(\overline{W_r}, \mathbb{Q}_p((2r+1)))$ defined in [BK], i.e. that the corresponding extension of Galois modules is crystalline. The subgroup H^1_f is the image of the Bloch-Kato exponential map, which is the connecting map in the long exact sequence in cohomology coming from the short exact sequence of Galois modules

$$0 \to W \to B_{\text{cris}}^{\phi=1} \otimes V \oplus \mathrm{Fil}^0 B_{\text{dR}} \otimes V \to B_{\text{dR}} \otimes V \to 0.$$

Because $D_{\text{cris}}(V)^{\phi=1} = 0$, the inverse of the exponential map induces a well-defined "logarithm" map

$$\log_{\text{BK}} : H^1(F_p, V) \to \frac{D_{\text{dR}}(V)}{\mathrm{Fil}^0}.$$

The element $\log_{\text{BK}}(\xi)$ lives in

$$\frac{D_{\text{dR}}(\epsilon H^{4r+1}(\overline{V}, \mathbb{Q}_\ell((2r+1)))}{\mathrm{Fil}^0} = \frac{\epsilon H^{4r+1}_{\text{dR}}(V)}{\mathrm{Fil}^{(2r+1)}}$$

By Poincare duality, the last space is identified with $\mathrm{Fil}^{2r+1} \epsilon H^{4r+1}_{\text{dR}}(V)^\vee$, which is $(M_{2r+2} \otimes \mathrm{Sym}^{2r} eH^1_{\text{dR}}(A_{/F_p}))^\vee$.

Definition VII.8. The p-adic Abel-Jacobi map

$$\mathrm{AJ}_p : \epsilon \mathrm{CH}^{2r+1}_0(W_{r/F_p}) \to (M_{2r+2}(F_p) \otimes \mathrm{Sym}^{2r} eH^1_{\text{dR}}(A_{/F_p}))^\vee.$$

sends a cycle Z to the Bloch-Kato logarithm of the extension class ξ_Z as in (7.1).

7.3 The case of weight two

In the case that $r = 0$, the variety W_r is just C, the projectors defined above are all trivial, and a homologically trivial cycle of codimension $2r + 1$ is a degree zero divisor. Write $\mathbb{T}_{\mathbb{Q}}$ for the Hecke algebra of level N^+. Then there is a projector $\epsilon_f \in \mathrm{Corr}(C) \otimes \mathbb{Q}$ which lies in the image of the map $\mathbb{T}_{\mathbb{Q}} \to \mathrm{Corr}(C) \otimes \mathbb{Q}$ and satisfies

$$\epsilon_f H^*_{\mathrm{dR}}(C/F) = F\omega_f \subset H^1_{\mathrm{dR}}(C/F).$$

Let Δ_0 be an arbitrary divisor on C. Then $\epsilon_f \Delta_0$ is automatically homologically trivial. The projector ϵ_f also gives an extension class attached to f using the Gysin sequence above, so it makes sense to apply the p-adic Abel-Jacobi map to $\epsilon_f \Delta_0$.

Remark VII.9. In this case (as well as the higher weight case) the p-adic Abel-Jacobi map can be constructed without reference to any projectors. In particular, in the weight two case, it can be identified with a formal group logarithm as follows: writing J for the Jacobian of C, there is, for each differential form $\omega \in \Omega_J$, a unique group homomorphism $\log_\omega : J(F_p) \to F_p$ with $dF_\omega = F_\omega$. If we pick an F_p-rational point of C to get a (classical) Abel-Jacobi map $C \to J$, we then get a map $C(F_p) \to F_p$, which coincides with AJ_p. (This map depends on our choice of classical Abel-Jacobi map, but the induced map on Pic^0 does not.)

CHAPTER VIII

Computation of the p-adic Abel-Jacobi map

8.1 The p-adic Abel-Jacobi map and Coleman integration

We will work with sums Δ of generalized Heegner cycles fibered above points $P_1, \ldots, P_m$; we assume this set to be Frobenius-stable. Write also $Q_1, \ldots Q_n$ for a choice of point on each supersingular residue disk. Write $S_P = \{P_1, \ldots, P_m\}$, $S_Q = \{Q_1, \ldots, Q_m\}$, and $S = S_P \cup S_Q$.

We will apply the formalism of Chapter VI to the affinoid $\mathcal{A} = C \backslash \bigcup_{P \in S} D(P, 1)$ and some choice of wide open neighborhood $\mathcal{W}$.

Proposition VIII.1. *If Δ is a sum of generalized Heegner cycles fibered above points in S_P, where the point P_i corresponds to the false elliptic curve A_i with level structure t_i, the generalized Heegner cycle Δ_i above P_i is given by the false isogeny $\phi_i : A \to A_i$, and $\alpha \in \operatorname{Sym}^{2r} eH^1_{dR}(A)$ is arbitrary, then*

$$AJ_F(\Delta)(\omega_f \wedge \alpha) = \sum_{P_i \in S_P} \langle F_f(P_i) \wedge \alpha, \mathrm{Cl}_{P_i}(\Delta_i) \rangle.$$

(Here, $\mathrm{Cl}_{P_i}(\Delta_i)$ denotes the image of Δ_i under the cycle class map attached to the fiber A_i^r of C above P_i, not the global cycle class map, which annihilates Δ by construction.)

Proof. This argument mimics the proof of Proposition 3.18 of [BDP] – in fact, it is strictly simpler, as that paper must deal with issues related to cusps of modular curves. To compute $AJ_F(\Delta)(\omega_f \wedge \alpha)$, we need to compute

$$\langle \log_{BK}(\xi_\Delta), \omega_f \wedge \alpha \rangle.$$

Here, ξ_Δ signifies the extension class D_Δ, which sits in the exact sequence of filtered Frobenius modules

$$0 \to H^1(C, \mathcal{L}_{2r,2r}, \nabla)(2r+1) \to D_\Delta \to F \to 0.$$

Explicitly, D_Δ is the set of pairs (η, β), where $\beta \in F$ and η is a cohomology class in

$$H^1(C \backslash S_P, \mathcal{L}_{2r,2r}, \nabla)(2r+1)$$

whose residue at each P_i is $\beta \, \mathrm{Cl}_{P_i}(\Delta)$.

To write down this logarithm, we must find a class $\eta_{\mathrm{hol}} \in \mathrm{Fil}^0 D_\Delta$ and a class $\eta_{\mathrm{frob}} \in (D_\Delta)^{\phi^{\deg(F_p/\mathbb{Q}_p)}=1}$, both mapping to 1 in F, then take their difference, which is well-defined up to $\mathrm{Fil}^{2r+1} H^1(C, \mathcal{L}_{2r,2r}, \nabla)$. We will think of η_{hol} and η_{frob} as classes in $H^1(C \backslash S_P, \mathcal{L}_{r,r}, \nabla)$, both required to have residue $\mathrm{Cl}_{P_i}(\Delta_i)$ at P_i.

One has

$$\mathrm{Fil}^0 H^1(C - S_P, \mathcal{L}_{2r,2r}, \nabla)(2r+1) = H^0(C \setminus S_P, \underline{\omega}^r) \otimes \mathrm{Sym}^{2r} eH^1_{\mathrm{dR}}(A).$$

In particular, η_{hol} is represented by an $\mathcal{L}_{r,r}$-valued one-form that is holomorphic away from S_P and has a simple pole at each $P_i \in S_P$ with residue Δ_i (or is holomorphic at P_i if $\Delta_i = 0$). Possibly enlarging S_P, we may assume the centers of the deleted disks include all the poles of η_{frob}.

To compute

$$\langle \log_{BK}(\xi_\Delta), \omega_f \wedge \alpha \rangle,$$

we need to pick primitives for $\omega \wedge \alpha$ in each disk, multiply by $\eta_{\mathrm{hol}} - \eta_{\mathrm{frob}}$, and sum the residues over the points in S. Now Lemma VI.12 tells us that if we pick the global Coleman primitive, then the contribution to the sum from S_Q cancels. Hence the sum simplifies to

$$\sum_{P_i \in S_P} \mathrm{res}_{P_i}(\eta, F_f \wedge \alpha) = \langle F_f(P_i) \wedge \alpha, \mathrm{cl}_{P_i}(\Delta)\rangle$$

(we are using the fact that $F_f \wedge \alpha$ is the Coleman primitive for $\omega_f \wedge \alpha$). □

The next proposition, which is proven as in [BDP] and only needed in the higher weight case, shows that we can move this result of the previous proposition from the various $P_i \in S$ to the point P_A corresponding to the fixed false elliptic curve A.

Proposition VIII.2 (BDP 3.21)**.** *If Δ_ϕ is supported over a single point P'_A, then we have*

$$AJ_F(\Delta_\phi)(\omega_f \wedge \alpha) = \langle \phi^* F_f(P_{A'}), \alpha\rangle_A.$$

where the pairing occurs on $e \operatorname{Sym}^{2r} H^1_{dR}(A)$.

8.2 Computing Coleman primitives for p-adic modular forms

In this section, we will use the following conventions, which are slightly different from those of [BDP]. A lowercase letter is a p-adic or locally analytic modular form, and the corresponding capital letter is its Serre-Tate-expansion, a power series in T (on some fixed ordinary residue disk). As in Chapter V, write θ for the operator on the space of locally analytic modular forms and $\Theta = (1+T)\frac{d}{dT}$ for the corresponding operator on power series; that is, if $g = G(T)\widehat{\omega}^k$ in a fixed residue disk of C, then

$$\theta g = (\Theta G)\widehat{\omega}^{k+2}$$

Because the main theorem of this section may be of use in other situations, we state it for arbitrary p-adic modular form f of weight $\rho + 2$ (for the fixed f from Chapter II, of course, $\rho = 2r$ is even). Consider f as a section ω_f of $\underline{\omega}^\rho \otimes \Omega \subseteq$

$\mathrm{Sym}^\rho e\mathcal{H}^1 \otimes \Omega$ using the Kodaira-Spencer map, and write g for the Coleman primitive of ω_f. In particular, g is a section of $\mathrm{Sym}^\rho e\mathcal{H}^1$ satisfying $\nabla g = \omega_f$.

In terms of the Serre-Tate basis for $\mathrm{Sym}^\rho e\mathcal{H}^1$ given by $\widehat{\omega}^{\rho-i}\widehat{\eta}^i$ for $i = 0, \ldots, \rho$, we may write

$$g = \sum_{i=0}^{\rho} G_i(T)\widehat{\omega}^{\rho-i}\widehat{\eta}^i. \tag{8.1}$$

(Here we are using the fact that g is rigid on residue disks.)

The formal power series $G_i(T)$ are actually the T-expansions of locally analytic modular forms of weight $2\rho - i$. To see this, recall that the $\mathcal{O}_X$-linear cup product pairing on $\mathcal{H}^1$ extends to a pairing on $\mathrm{Sym}^\rho e\mathcal{H}^1$ by the rule

$$\langle \alpha_1 \otimes \ldots \otimes \alpha_\rho, \beta_1 \otimes \ldots \otimes \beta_\rho \rangle = \frac{1}{\rho!} \sum_{\sigma \in S_\rho} \prod_i \langle \alpha_i, \beta_{\sigma i} \rangle \tag{8.2}$$

Following [BDP], we define a locally analytic modular form $\widetilde{g}_i$ by the rule

$$\widetilde{g}_i(A, t) = \langle g(A, t), \omega^i \eta^{\rho-i} \rangle \omega^{2\rho-i},$$

where $\omega \in \underline{\omega}(D)$ and $\eta \in \mathcal{H}^1(D)$ are chosen with $\langle \omega, \eta \rangle = 1$. (Replacing ω by $\lambda\omega$ has the effect of replacing η by $\lambda^{-1}\eta$, so the form does not depend on any choices.)

Combining (8.1) and (8.2) shows that the Serre-Tate expansion of $\widetilde{g}_i$ is given by

$$\widetilde{g}_i = \frac{(-1)^i}{\binom{\rho}{i}} G_i(T)\widehat{\omega}^{2\rho-i}. \tag{8.3}$$

Note that $\widetilde{g}_i$ is a locally analytic modular form on all of $\mathcal{X}$, not just on the ordinary locus (where its T-expansions make sense and where formula (8.3) holds).

These components can be computed by inverting the differential operator θ:

Theorem VIII.3. *One has*

$$\widetilde{g}_i^\flat = i!\theta^{-1-i} f^\flat.$$

In particular, $\widetilde{g}_i^\flat$, which is a priori only locally analytic, is a p-adic modular form, i.e., is rigid on the ordinary locus.

Proof. The theorem is equivalent to the statement that

$$\theta^{1+i}\widetilde{g_i} = i!f$$

(the flat operator arises upon inverting θ). It suffices to show that $\theta\widetilde{g_0} = f$ and $\theta\widetilde{g_i} = i\widetilde{g_{i-1}}$ for $i > 0$.

Using the Leibniz rule and Katz's computation of ∇ on the basis $\{\widehat{\omega}, \widehat{\eta}\}$ yields

$$\begin{aligned}\nabla(G_i\widehat{\omega}^{\rho-i}\widehat{\eta}^i) &= G_i'\widehat{\omega}^{\rho-i}\widehat{\eta}^i \otimes dT + G_i\nabla(\widehat{\omega}^{\rho-i}\widehat{\eta}^i) \\ &= \Theta G_i\widehat{\omega}^{\rho-i}\widehat{\eta}^i \otimes \operatorname{dlog} q + (\rho - i)G_i\widehat{\omega}^{\rho-i-1}\widehat{\eta}^{i+1} \otimes \operatorname{dlog} q,\end{aligned}$$

Summing this equality over i and reindexing gives

$$\nabla g = \Theta G_0\widehat{\omega}^\rho \otimes \operatorname{dlog} q + \sum_{i=1}^{\rho} (\Theta G_i + (\rho - i + 1)G_{i-1})\, \widehat{\omega}^{\rho-i}\widehat{\eta}^i \otimes \operatorname{dlog} q$$

On the other hand, since g is a primitive,

$$\begin{aligned}\nabla g &= KS(f) \\ &= KS(F\widehat{\omega}^{\rho+2}) \\ &= F\widehat{\omega}^\rho \otimes \operatorname{dlog} q.\end{aligned}$$

It follows that $\Theta G_0 = F$ and that $\Theta G_i = -(\rho - i + 1)\ G_{i-1} = 0$ for $i < \rho$. The result now follows from (8.3). $\square$

In the weight two case ($r = 0$), the operator $\mathcal{O}_C \to \underline{\omega}^{\otimes 2}$ is just d, followed by the Kodaira-Spencer isomorphism. The content of the above proposition is then that the limit defining θ^{-1} exists (and that the Coleman primitive is a primitive).

Consider the particular f fixed in Chapter II, so that $\rho = 2r$, and write g_j for the jth component of the Coleman primitive as before. The following proposition, which is proven using Proposition VIII.2 as in BDP, relates the components $\widetilde{g}_i$ to the p-adic Abel-Jacobi map:

Proposition VIII.4 (BDP 3.22)**.** *Write d for the degree of the false isogeny $A \to A'$. Then*

$$AJ_F(\Delta_\phi)(\omega_f \wedge \omega^j \eta^{2r-j}) = d^j g_j(A', t', \omega').$$

Lemma VIII.5. *Suppose that the weight of f is 2. Then for any zero-cycle Δ on C, one has*

$$(\theta^{-1} f^\flat)(\epsilon_f \Delta_0) = (\theta^{-1} f^\flat)(\Delta_0)$$

Proof. Because the operators U and V commute with all the operators T_ℓ, the p-adic modular form $f^\flat$ is still an eigenform with the same Hecke eigenvalues as f away from p. It follows from Proposition III.6 and an approximation argument that

$$(\theta^{-1} f^\flat)|_{T_\ell} = \ell^{-1} \theta(f^\flat|_{T_\ell}) = a_\ell \ell^{-1} \theta f^\flat$$

Write $T_\ell^* \Delta_0$ for the zero-cycle on C given by the Hecke orbit of Δ_0. Thus for g a modular function we have $g|_{T_\ell}(P) = \frac{1}{\ell} g(T_\ell^* P)$. But then

$$\begin{aligned}(\theta^{-1} f^\flat)(T_\ell^* \Delta_0) &= \ell(\theta^{-1} f^\flat)|_{T_\ell}(\Delta_0) \\ &= a_\ell(\theta^{-1} f^\flat)(\Delta_0)\end{aligned}$$

Write $\lambda_f : \mathbb{T} \to F$ for the homomorphism attached to the newform f. Then $\lambda_f(T_\ell) = a_\ell$, so the above computation shows that for an arbitrary $T \in \mathbb{T}$ one has

$$(\theta^{-1} f^\flat)(T^* \Delta_0) = \lambda_f(T)(\theta^{-1} f^\flat)(\Delta_0).$$

By design $\lambda_f(\epsilon_f) = 1$, so we are done. □

CHAPTER IX

The p-adic L-function

9.1 Spaces of Hecke characters

This subsection closely follows the notation of [BDP]. The p-adic L-function we study interpolates special values of the Rankin-Selberg L-function $L(f, \chi, s)$ as χ varies over the space of unramified Hecke characters of K. Since K satisfies the Heegner hypothesis for N^+, there is an ideal $\mathfrak{N}^+$ of K with norm N^+. As in the introduction, we single out a few spaces of Hecke characters on K (the motivation for these choices is in Section 1.1.4). A Hecke character χ with infinity type (ℓ_1, ℓ_2) is *critical* for f if one of the following conditions holds:

- (The type 1 case): $1 \leq \ell_1, \ell_2 \leq k-1$.
- (The first type 2 case): $\ell_1 \geq k$ and $\ell_2 \leq 0$.
- (The second type 2 case): $\ell_1 \leq 0$ and $\ell_2 \geq k$.

One says χ is *central critical* if in addition $\ell_1 + \ell_2 = k$ and the central character of χ matches the nebentypus of f. We will write $\Sigma_{\mathrm{cc}}^{(1)}$ for the set of central critical characters of type 1 and $\Sigma_{\mathrm{cc}}^{(2)}$ for the set of central critical characters in the first type 2 case. Because the values of critical Hecke characters are algebraic, we may view them as p-adic numbers via our fixed embedding. As is explained in the discussion before Remark 5.8 of [BDP], the set $\Sigma_{\mathrm{cc}}^{(2)}$ comes with a p-adic topology (it is the topology of uniform convergence on compact subsets, where we think of $\Sigma_{\mathrm{cc}}^{(2)}$ as a subset of the space of functions from the prime-to-p ideles of K to $\mathcal{O}_{\mathbb{C}_p}$).

Write $\widehat{\Sigma_{\mathrm{cc}}^{(2)}}$ for the completion of $\Sigma_{\mathrm{cc}}^{(2)}$ with respect to this topology. Write h for the class number of K; for each integer t, there is a Hecke character ψ_t given by the rule

$$\psi_t(\mathfrak{a}) = a^{6t}/\overline{a}^{6t}$$

where $(a) = \mathfrak{a}^h$. Note that the infinity type of ψ_t is $(6h, -6h)$. It follows that $\chi\psi_t$ is central critical of type 2 for χ central critical of any type (and t large) and $\chi\psi_{p^n(p-1)} \to \chi$ as $n \to \infty$. It follows that we may view $\Sigma_{\mathrm{cc}}^{(1)}$ as a subset of $\widehat{\Sigma_{\mathrm{cc}}^{(2)}}$.

9.2 The Waldspurger-type result

Using the fixed complex structure J_τ on $M_2(\mathbb{R})$, thought of as a map

$$J_\tau : M_2(\mathbb{R}) \to \mathbb{C}^2,$$

we get a differential form $\omega_{\mathbb{C}} = J_\tau^*(2\pi i dz_1)$ on $M_2(\mathbb{R})$ (holomorphic for this complex structure). Abusing notation, also write $\Omega_{\mathbb{C}}$ for the corresponding form on $B \otimes \mathbb{R}$ (which is really $\iota_\infty^* \omega_{\mathbb{C}}$).

There is a bijective correspondence between $\iota_\infty(\mathcal{O})$-stable sublattices of $\mathbb{C}^2$ and pairs (A, ω) of a false elliptic curve over $\mathbb{C}$ and a section of $e\Omega_{A/\mathbb{C}}$. To a pair (A, ω) we attach the lattice

$$\mathcal{O}\left\{ \int_\gamma \omega \middle| \gamma \in eH_1(A) \right\},$$

and to an $\mathcal{O}$-stable lattice Λ we assign the false elliptic curve $\mathbb{C}^2/\Lambda$ together with the form $2\pi i dz_1$.

Again using the complex structure J_τ on $M_2(\mathbb{R})$, we may view a modular form g as a function on pairs (Λ, t), where Λ is an $\mathcal{O}$-stable sublattice of $B \otimes \mathbb{R}$ and t is an element of exact order N^+ in $\frac{B\otimes\mathbb{R}}{\Lambda}$. Explicitly, this function is given by the rule

$$g(\Lambda, t) = g\left(\frac{B\otimes\mathbb{R}}{\Lambda}, t, \omega_{\mathbb{C}}\right).$$

Scaling the lattice Λ by some $\lambda \in \mathbb{C}$ corresponds to multiplying the period integral by λ, so the corresponding one-form $\omega_{\mathbb{C}}$ is divided by λ. Hence for g of nebentypus ϵ_g we have

$$g(\lambda\Lambda, \lambda t) = \lambda^{-k}\epsilon_g(\lambda)g(\Lambda, t).$$

Fix a generator $t \in (\mathfrak{N}^+)^{-1}$ of the cyclic group

$$\mathbb{Z}/N = (\mathfrak{N}^+)^{-1}/\mathcal{O}_K.$$

Then $\iota_\tau(t)$ is an element of exact order N^+ in $\frac{B\otimes\mathbb{R}}{\mathcal{O}_B}$.

More generally, given a fractional ideal $\mathfrak{a}$ of K prime to N^+, there is a canonical generator $t_{\mathfrak{a}}$ of

$$\frac{\mathfrak{N}^{+^{-1}}\mathfrak{a}^{-1}}{\mathfrak{a}^{-1}},$$

as described before Lemma 4.5 of [BDP]. To define it, pick $\alpha \equiv 1 \mod \mathfrak{N}^+$ such that the ideal $\mathfrak{b} = \alpha\mathfrak{a}$ is contained in $\mathcal{O}$; then $t_{\mathfrak{a}}$ is the image of t under the composition

$$\frac{\mathfrak{N}^{+^{-1}}}{\mathcal{O}_K} \to \frac{\mathfrak{N}^{+^{-1}}\mathfrak{b}^{-1}}{\mathfrak{b}^{-1}} \xrightarrow{\cdot\alpha} \frac{\mathfrak{N}^{+^{-1}}\mathfrak{a}^{-1}}{\mathfrak{a}^{-1}}.$$

If $\mathfrak{a}$ is an ideal of $\mathcal{O}_K$, then $\iota_\tau(t_{\mathfrak{a}})$ is an element of exact order N^+ in $\frac{B\otimes\mathbb{R}}{\mathfrak{a}_B{}^{-1}}$, using the notation of Section 2.6.1. Abusing notation, we will write $t_{\mathfrak{a}}$ rather than $\iota_\tau(t_{\mathfrak{a}})$ for this element.

Lemma IX.1. *Let $\mathfrak{a}$ be an ideal prime to N and let χ be a central critical Hecke character of infinity type $(k+j, -j)$. Then for any t, the expression*

$$\chi^{-1}(\mathfrak{a})\mathbf{N}\mathfrak{a}^{-j}\delta_k^j f(\mathfrak{a}_B^{-1}, t_{\mathfrak{a}})$$

only depends on the class of $\mathfrak{a}$ in $\mathrm{Cl}(\mathcal{O})$.

Proof. Scaling the pair $(\mathfrak{a}, t)$ by $\lambda \in K$, we get

$$\delta_k^j f(\lambda^{-1}\mathfrak{a}_B^{-1}, t_{\lambda\mathfrak{a}}) = \epsilon_f(\lambda)\lambda^{k+2j} f(\mathfrak{a}, t).$$

On the other hand,

$$\chi^{-1}(\lambda\mathfrak{a}) = \epsilon_\chi^{-1}\lambda^{-k-j}\overline{\lambda}^j \chi^{-1}(\mathfrak{a})$$

and

$$\mathbf{N}(\lambda\mathfrak{a})^{-j} = \lambda^{-j}\overline{\lambda}^{-j}(\mathbf{N}\mathfrak{a})^{-j}.$$

The result follows from the assumption $\epsilon_f = \epsilon_\chi$. $\square$

The following result is Theorem 3.2 of [Pr]:

Theorem IX.2. *Let χ be an unramified Hecke character of K of infinity type $(k + j, -j)$ whose central character is the nebentypus of f. Then one has, for some $\alpha(f, f_{\mathrm{GL}_2}) \in K$*

$$C(f, \chi)L(f, \chi^{-1}, 0) = \alpha(f, f_{\mathrm{GL}_2}) \left| \sum_{\mathfrak{a} \in \mathrm{Cl}(\mathcal{O}_K)} \chi^{-1}(\mathfrak{a})\mathbf{N}\mathfrak{a}^{-j} \cdot (\Theta_{\mathbb{R}}^j f)(\mathfrak{a}_B^{-1}, t_\mathfrak{a}) \right|^2,$$

where

$$C(f, \chi) = \frac{1}{4}\pi^{k+2j-1}\Gamma(j + 1)\Gamma(k + j)w_K\sqrt{|d_K|}2^{\#S_f} \prod_{\ell | N^-} \frac{\ell - 1}{\ell + 1}$$

and S_f is the set of primes which ramify in K that divide N^+ but do not divide the conductor of the Nebentypus of f.

The element $\alpha(f, f_{\mathrm{GL}_2})$ is the quotient of the Petersson inner products

$$\frac{\langle f_{\mathrm{GL}_2}, f_{\mathrm{GL}_2}\rangle}{\langle f, f\rangle}.$$

Because of our normalization of f, it is an element of K by a theorem of Harris and Kudla. By Theorem 2.4 of [Pr], it is integral at p provided $p > k + 1$ and $p \nmid \prod_{\ell | N}(\ell - 1)(\ell)(\ell + 1)$.

9.2.1 CM points and CM triples

This section eliminates the absolute value signs that occur in the statement of Theorem IX.2 by comparing the complex conjugation action on the space of modular forms with an Atkin-Lehner involution. Fix for now a primitive N^+th root of unity $\zeta \in \overline{\mathbb{Q}}$, i.e. a trivialization of μ_{N^+}. (We will make a particular choice later.) Suppose L is some field containing K, A'/L is a false elliptic curve with normalized CM by $\mathcal{O}_K$, and $P = eP$ is a torsion point of exact order N^+ on A'. Then there is a point of $C(F)$ given by the false elliptic curve A' together with the level structure

$$\mu_{N^+} \times \mu_{N^+} \approx \mathbb{Z}/N^+ \oplus \mathbb{Z}/N^+ \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix} \mapsto P} A'[\mathcal{N}].$$

Such a point will be denoted (A, P).

A *CM triple* over L is an isomorphism class of triple (A, P, ω) where $\omega \in e\Omega_{A/L}$ is nonvanishing. Using the above formalism, one can think of a CM triple as a point on the underlying space of the bundle $\underline{\omega}_L$.

There is an action $\star$ of $\mathrm{Cl}(\mathcal{O}_K)$ on the set of CM triples, given by the rule

$$\mathfrak{a} \star (A', P, \omega) = (A'/A'[\mathfrak{a}], P_0, \omega_0)$$

where P pushes forward to P_0 and ω_0 pulls back to ω.

Assuming also that $\sqrt{-N^+} \in L$, there is an Atkin-Lehner involution, denoted w_{N^+}, on the underlying space of the bundle $\underline{\omega}_L$ (it is not an automorphism of line bundles, but rather lies over an involution on C, which we also call an Atkin-Lehner involution and also write w_{N^+}). It is described by the following rule:

$$(A', P, \omega) \mapsto (A'/P, P', \sqrt{-N^+}\omega),$$

where $P' = eP'$ is chosen so that the Weil pairing $(\mathrm{Image}(P), P') = \zeta$.

There is a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ semilinear complex conjugation action on $C_{\mathbb{C}}$ and the underlying space of $\underline{\omega}_{\mathbb{C}}$, given on arbitrary points (not just CM triples) by

$$\overline{(A', t, \omega)} = (A'^{\sigma}, t^{\sigma}, \omega^{\sigma}),$$

where σ denotes base change along the nontrivial map $\mathbb{C} \to \mathbb{C}$. (Note that even if A' and A'^{σ} are isomorphic as abelian surfaces, they will not be isomorphic as false elliptic curves.)

By Lemma 5.2 of [BDP], the compatibility of the Atkin-Lehner involution with the operation of $\mathrm{Cl}(K)$ on the set of CM triples is given over $\mathbb{C}$ by the rule

$$\mathfrak{a} \star w_N(A', P, \omega_{\mathbb{C}}) = w_N(\mathfrak{a} \star (A', \mathbf{N}\mathfrak{a}^{-1}P, \omega_{\mathbb{C}})) \tag{9.1}$$

In the course of establishing the following proposition, we will fix a particular ζ, which depends on the fixed CM elliptic curve A; from now on, any reference to an Atkin-Lehner involution is with respect to this ζ.

Proposition IX.3. *There exists an ideal $\mathfrak{b}$ of $\mathcal{O}_K$, and a scalar $b_{N^+} \in \mathcal{O}_K$, with the property that for any CM triple, one has*

$$\overline{(A', P, 2\pi i dz)} = \mathfrak{b} \star w_{N^+}(A, P, \frac{b_{N^+}}{\sqrt{-N^+}} 2\pi i dz)$$

Proof. Because A^{σ} has false endomorphisms by $\mathcal{O}_K$, there is a false isogeny $A \to A^{\sigma}$ whose kernel is of the form $A[\mathfrak{f}]$ for some ideal $\mathfrak{f}$ of K. If necessary, multiply by a scalar to ensure $(\mathfrak{f}, \mathfrak{N}^+) = 1$. Pick the ideal $\mathfrak{b}$ to be prime to $\mathfrak{f}\mathfrak{N}^+$ and to satisfy

$$\mathfrak{b}\mathfrak{N}^+\mathfrak{f}^{-1} = (b_{N^+}).$$

for some scalar b_{N^+}. Then multiplication by b_{N^+}, followed by the natural projection, gives an identification

$$\frac{A[N^+]}{A[\mathfrak{N}^+]} \to A^{\sigma}[\overline{\mathfrak{N}^+}].$$

In particular, one may lift $\overline{P}$ to $P' = eP' \in A[N]$. Set $\zeta = (P, P')$. The result is now plain from formula 9.1. $\square$

As in the modular curve case, there is an involution $g \mapsto g_\rho$ on the space of weight k modular forms for $C/\mathbb{C}$ by the rule

$$g_\rho(A, t, \omega) := \overline{g(\overline{A, t, \omega})}.$$

Lemma IX.4. *If g is an eigenform with $T_\ell g = a_\ell g$, then g_ρ is an eigenform with $T_\ell g_\rho = \overline{a_\ell} g_\rho$.*

Proof. One has

$$\begin{aligned} g_\rho|_{T_\ell}(A, t, \omega) &= \overline{\frac{1}{\ell}\sum_{i=0}^{\ell} g(A^\sigma/C_i^\sigma, t_i^\sigma, \omega_i^\sigma)} \\ &= \overline{g_{T_\ell}(A^\sigma, t^\sigma, \omega^\sigma)} \\ &= \overline{a_\ell}\,\overline{g(A^\sigma, t^\sigma, \omega^\sigma)} \\ &= \overline{a_\ell} g_\rho(A^\sigma, t^\sigma, \omega^\sigma) \end{aligned}$$

$\square$

A similar computation shows that if g has nebentypus ϵ_g, then g_ρ has nebentypus $\overline{\epsilon_g}$. It follows as in the proof of Lemma 5.2 of [BDP] that there is a complex scalar W_g of norm one, depending only on g and our choice of ζ, such that

$$g_\rho(w_N(E, P, \omega)) = W_g g(E, t, \omega).$$

For $\chi \in \Sigma^{(2)}$, set

$$W(f, \chi) = W_f \epsilon_f(\mathbf{N}\mathfrak{b})^{-1}\chi_j(\mathfrak{b})(-N)^{k/2+j} b_N^{-k-2j}.$$

Abbreviate $\chi\mathbf{N}^{-j}$ as χ_j. Then one has:

Proposition IX.5. *Under the hypotheses of Theorem IX.2, one has*

$$C(f,\chi)L(f,\chi^{-1},0) = \alpha(f, f_{\mathrm{GL}_2})W(f,\chi)\left(\sum_{\mathfrak{a}\in \mathrm{Cl}(\mathcal{O}_K)} \chi^{-1}(\mathfrak{a})\mathbf{N}\mathfrak{a}^{-j}\cdot(\Theta_{\mathbb{R}}^{j}f)(\mathfrak{a}\star(A,P,\omega_{\mathbb{C}}))\right)^2,$$

Proof. The formula

$$\overline{\chi_j^{-1}(\mathfrak{a})\Theta_{\mathbb{R}}^{j}f(\mathfrak{a}\star(A,P,\omega_{\mathbb{C}})} = w_f(-N)^{k/2+j}b_{N^+}^{-k-2j}\chi_j(\mathfrak{b})\epsilon_f(\mathbf{N}\mathfrak{b})^{-1}\chi_j(\bar{\mathfrak{a}}\mathfrak{b})^{-1}\Theta_{\mathbb{R}}f(\bar{\mathfrak{a}}\mathfrak{b}\star(A,P,\omega_C))$$

is established in the proof of Theorem 5.4 of [BDP], except with Remark (1) in that proof replaced by Proposition IX.3 above. The result follows from summing this formula over $\mathfrak{a}$, using Theorem IX.2. □

The following lemma expresses the operator $\flat$ on the space of locally analytic p-adic modular forms in terms of the action of $\mathrm{Cl}(K)$ on CM triples.

Lemma IX.6. *If g is a locally analytic p-adic modular form of integer weight k satisfying*

$$T_p g = b_p g$$

and

$$\langle p\rangle = \epsilon(p)g$$

for some character ϵ, and (A',t,ω') is a CM triple, then

$$g^{\flat}(A',t,\omega') = g(A',t',\omega') - \frac{\epsilon(p)b_p}{p^k}g(\mathfrak{p}\star(A',t',\omega')) + \frac{\epsilon(p)}{p^{k+1}}g(\mathfrak{p}^2\star(A',t',\omega'))$$

Proof. This computation is the same as that of Lemma 3.23 of [BDP]: since A' has complex multiplication, the canonical subgroup of A' is $A'[\mathfrak{p}]$. Thus $g|_V(A',t',\omega') = \mathfrak{p}\star(A',p^{-1}t',p\omega')$ and $g|_{[p]V^2} = \mathfrak{p}^2\star(A',p^{-1}t',p\omega')$. The result then follows from

$$VU - UV = 1 - T_pV + \frac{1}{p}[p]V^2.$$

□

9.3 The p-adic L-function

Recall the fixed non-vanishing global section ω of the line bundle $e\Omega_{A/H}$ on A, defined over the Hilbert class field of K. Define a period $\Omega \in \mathbb{C}$ by the rule

$$\omega = \Omega\omega_C.$$

Define also a p-adic period $\Omega_p \in \mathbb{C}_p$ by the rule

$$\omega = \Omega_p\widehat{\omega},$$

where $\widehat{\omega}$ is the formal section picked in Chapter V (which depended upon a choice of basis for $eT_p\widetilde{A}$, where $\widetilde{A}$ denotes the reduction of A mod $\mathfrak{p}$).

Proposition IX.7. *For $\chi \in \Sigma^2_{cc}(\mathcal{N})$ of infinity type $(k+j,-j)$, the quantity*

$$L_{alg}(f,\chi^{-1},0) := \alpha(f,f_{\mathrm{GL}_2})^{-1}W(f,\chi)^{-1}C(f,\chi,c)L(f,\chi,0)/\Omega^{(2(k+2j))}$$

belongs to $\overline{\mathbf{Q}}$, and is computed by the formula

$$L_{alg}(f,\chi^{-1},0) = \left(\sum_{[\mathfrak{a}]\in\mathrm{Cl}(\mathcal{O})} \chi_j^{-1}(\mathfrak{a})\cdot\Theta^j_\infty f(\mathfrak{a}\star(A,t,\omega))\right)^2$$

The following corollary then follows immediately from the coincidence of the values of the forms θf and $\Theta_{\mathbb{R}} f$ on CM points.

Corollary IX.8. *For $\chi \in \Sigma^{(2)}_{cc}$ of infinity type $(k+j,-j)$,*

$$L_{alg}(f,\chi^{-1},0) = \left(\sum_{[\mathfrak{a}]\in\mathrm{Cl}(\mathcal{O}_K)} \chi_j^{-1}(\mathfrak{a})\cdot\theta^j f(\mathfrak{a}\star(A,t,\omega))\right)^2.$$

Now set

$$L_p(f,\chi) = \Omega_p^{2(k+2j)}(1-\chi^{-1}(\overline{\mathfrak{p}})a_p + \chi^{-2}(\overline{\mathfrak{p}})\epsilon_f(p)p^{k-1})^2 L_{\mathrm{alg}}(f,\chi^{-1},0).$$

The following proposition expresses the Euler factor dropped at $\mathfrak{p}$ in terms of the operator $\flat$ on the space of p-adic modular forms; the computation is in Theorem 5.9 of [BDP]. (Note that we have replaced the algebraic form ω with the p-adic form $\widehat{\omega}$.)

Proposition IX.9. *For $\chi \in \Sigma_{cc}^{(2)}$ of infinity type $(k+j,-j)$, one has*

$$L_p(f,\chi) = \left(\sum_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_K)} \chi_j^{-1}(\mathfrak{a}) \cdot \theta^j f^\flat(\mathfrak{a} \star (A,t,\widehat{\omega})) \right)^2$$

9.4 Special values of L_p

This section investigates the properties of L_p outside the range of interpolation.

Proposition IX.10. *The function $\chi \mapsto L_p(f,\chi)$ extends to a continuous function on all of $\widehat{\Sigma_{cc}(\mathcal{N})}$ (which we will still write as L_p.)*

Proof. If two characters χ_1 and χ_2 are sufficiently close in the topology on $\widehat{\Sigma^2_{\mathrm{cc}}(\mathcal{N})}$, then their infinity types satisfy the congruence

$$j_1 \equiv j_2 (\mathrm{mod}\ (p-1)p^{M-1}).$$

(to see this, evaluate on ideles congruent to 1 mod $\mathcal{N}$). It follows from Proposition V.15 that

$$\theta^{j_1} f(A,t,\widehat{\omega}) \equiv \theta^{j_2} f^\flat(A,t,\widehat{\omega}) \mod p^M$$

at any ordinary CM point. The result follows from the formula of Theorem 9.4, which computes the value of L_p in terms of values of $f^\flat$ at ordinary CM points; moreover, it follows that the formula of theorem computes the values of L_p for any $\chi \in \widehat{\Sigma^{(2)}_{\mathrm{cc}}}$. $\square$

The following theorem is the main result of this document.

Theorem IX.11. *Suppose χ is a central critical character with infinity type $(k-1-j, 1+j)$, with $0 \le j \le 2r$. Then*

$$\frac{L_p(f,\chi)}{\Omega_p^{2(2r-2j)}} = (1-\chi^{-1}(\overline{\mathfrak{p}})a_p + \chi^{-2}(\overline{\mathfrak{p}})\epsilon_f(p)p^{k-1})^2 \left(\sum_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_K)} \chi^{-1}(\mathfrak{a})\mathbf{N}(\mathfrak{a}) \cdot AJ_F(\Delta_{\phi_{\mathfrak{a}}})(\omega_f \wedge \omega_A^j \eta_A^{2r-j}) \right)^2$$

Proof. The proof of the preceding proposition establishes the formula

$$L_p(f,\chi) = \left(\sum_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_K)} \chi_{-1-j}^{-1}(\mathfrak{a}) \cdot \theta^{-1-j} f^\flat(\mathfrak{a} \star (A,t,\widehat{\omega})) \right)^2 .$$

By definition of Ω_p we have (using that the weight of $\theta^{-1-j}f^\flat$ is $k-2-2j$) that

$$\frac{L_p(f,\chi)}{\Omega_p^{2(2r-2j)}} = \left(\sum_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_K)} \chi_{-1-j}^{-1}(\mathfrak{a}) \cdot \theta^{-1-j} f^\flat(\mathfrak{a} \star (A,t,\omega)) \right)^2 .$$

Lemma VIII.3 shows that the value of θ^{-1-j} acting on $f^\flat$ is $\frac{1}{j!}\widetilde{g_j}^\flat$ where $\widetilde{g_j}$ denotes the jth component of the Coleman primitive for f, which gives

$$\frac{L_p(f,\chi)}{\Omega_p^{2(2r-2j)}} = \frac{1}{j!}\left(\sum_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_K)} \chi_{-1-j}^{-1}(\mathfrak{a}) \cdot \theta^{-1-j} \widetilde{g_j}^\flat(\mathfrak{a} \star (A,t,\omega)) \right)^2 .$$

By Lemma IX.6 (and a rearrangement of the sum), one can remove the operator $\flat$ on $\widetilde{g_j}$ by dropping an Euler factor:

$$\frac{L_p(f,\chi)}{\Omega_p^{2(2r-2j)}} = (1-\chi^{-1}(\overline{\mathfrak{p}})a_p+\chi^{-2}(\overline{\mathfrak{p}})\epsilon_f(p)p^{k-1})^2\frac{1}{j!}\left(\sum_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_K)} \chi_{-1-j}^{-1}(\mathfrak{a}) \cdot \theta^{-1-j} \widetilde{g_j}(\mathfrak{a} \star (A,t,\omega)) \right)^2 .$$

Now apply Lemma VIII.4 to the Heegner isogeny $\phi_{\mathfrak{a}}$, of degree $(\mathbf{N}\mathfrak{a})^2$, to attain the final result. $\square$

BIBLIOGRAPHY

BIBLIOGRAPHY

[Be] Besser, Amnon. *CM cycles over Shimura curves.* J. Algebraic Geom. 4 (1995), no. 4, 659-691.

[BK] Bloch, Spencer and Kato, Kazuya. L-functions and Tamagawa numbers of motives. *The Grothendieck Festschrift, Vol. I,* 333-400, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990.

[BL] Birkenhake, Christina and Lange, Herbert. *Complex abelian varieties.* Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin (2004).

[BDP] Bertolini, M., Darmon, H., and Prasanna, K., *Generalised Heegner cycles and p-adic Rankin L-series*, Duke Math Journal, to appear.

[BDP2] Bertolini, M., Darmon, H., and Prasanna, K., *p-adic L-functions and the coniveau filtration on Chow groups.*, in progress.

[Br] Brakočević, Miljan. *Anticyclotomic p-adic L-function of central critical Rankin-Selberg L-value.* Int. Math. Res. Not. 2011, no. 21.

[Bu] Buzzard, Kevin. *Integral models of certain Shimura curves.* Duke Math. J. 87 (1997), no. 3, 591-612.

[Co] Coleman, Robert. *A p-adic Shimura Isomorphism and p-adic periods of modular forms*, Contemporary Mathematics, 165, 1994, 21-51.

[Co2] Coleman, Robert. *Reciprocity laws on curves*, Compositio Math. 72 (1989), no. 2, 205-235.

[Co3] Coleman, Robert. *Torsion Points on Curves and p-Adic Abelian Integrals.* Ann. of Math (2) 121 (1985), no. 1, 111-168.

[CS] Coates, J., and Sujatha, R. *Cyclotomic fields and zeta values.* Springer Monographs in Mathematics. Springer-Verlag, Berlin (2006).

[De] Deligne, P. *Equations différentielles à points singuliers réguliers.* Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York (1970).

[Di] Disegni, D. *p-adic Heights of Heegner Points on Shimura Curves.* Unpublished, available at http://arxiv.org/abs/1304.5204 (2013).

[Iw] Iwasawa, K. *Lectures on p-adic L-functions.* Ann. of Math. Studies, Princeton University Press, Princeton. (1972).

[Ka] Katz, N. p-adic properties of modular schemes and modular forms. *Modular functions of one variable, III* (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 69-190. Lecture Notes in Mathematics, Vol. 350, Springer, Berlin, 1973.

[Ka2] Katz, N. *Serre-Tate Local Moduli.* Algebraic Surfaces (Orsay 1976-1978), pp. 138-202, Lecture Notes in Math., 868, Springer, Berlin-New York (1981).

[Ka3] Katz, N. *p-adic L-functions for CM fields.* Invent. Math. 49, (1978), 99-297

[Kas] Kassaei, Payman. p-adic modular forms over Shimura Curves over $\mathbb{Q}$. Ph. D. thesis. Available online.

[KL] Kubota, T. and Leopoldt, H. *Eine p-adische Theorie der Zetawert. I. Einführung der p-adischen Dirichletschen L-Funktionen.* J. Reine Agnew. Math. 214/215 (1964) 328-339.

[KM] Katz, N. and Mazur, B. *Arithmetic moduli of elliptic curves.* Annals of Mathematics Studies, 108. Princeton University Press, Princeton, NJ, 1985.

[KO] Katz, N., and Oda, T. On the differentiation of de Rham cohomology classes with respect to parameters. J. Math. Kyoto Univ. 8 (1968), 199-213.

[GZ] Gross, B. and Zagier, D., *Heegner points and derivatives of L-series*, Invent. Math. 84 (1986), no. 2, 225-320.

[Ma] Masdeu, Marc. *CM cycles on Shimura curves, and p-adic L-functions.* Compositio Mathematica, volume 148, issue 04 (2013), pp. 1003-1032.

[Mo] Mori, Andrea. *Power series expansions of modular forms and their interpolation properties.* Int. J. Number Theory 7 (2011), no. 2, 529-77.

[Morita] Morita, Y. Reduction modulo p of Shimura curves. Hokkaido Mathematical Journal, 10 (2) (1981)

[Ne1] Nekovář, Jan. *p-adic Abel-Jacobi maps and p-adic heights.* In: The Arithmetic and Geometry of Algebraic Cycles (Banff, Canada, 1998), 367 - 379, CRM Proc. and Lect. Notes 24, Amer. Math. Soc., Providence, RI, 2000.

[Ne2] Nekovář, Jan. *On the p-adic height of Heegner cycles.* Math. Ann. 302 (1995), 609 - 686.

[Pr] Prasanna, K. *Integrality of a ratio of Petersson norms and level-lowering congruences.* Ann. of Math. (2) 163 (2006), no. 3, 901-967.

[PR] Perrin-Riou, Bernadette. *Points de Heegner et dérivées de fonctions L p-adiques*, Invent. Math. 89 (1987), no. 3, 455-510.

[Ru] Rubin, Karl. *p-adic L-functions and rational points on elliptic curves with complex multiplication.* Invent. math. 107 (1992), no. 2, 323-350.

[Sch] Scholl, A. *Motives for modular forms*, Invent. Math. 100 (1990), no. 2, 419-430.

[Se] Serre, J.-P. *Formes modulaires et fonctions zêta p-adiques. Modular functions of one variable, III* (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 191-268. Lecture Notes in Mathematics, Vol. 350, Springer, Berlin, 1973.

[Sh] Shimura, Goro. *Construction of Class Fields and Zeta Functions of Algebraic Curves.* Ann. of Math. (2) 85 (1967), no. 1, 58-159.

[Sk] Skinner, C. "A converse to a theorem of Gross-Zagier-Kolyvagin." Online lecture, http://video.ias.edu/jointiaspu/1213/0404-ChristopherSkinner (2013).

[Vi] *Arithmétique des algèbres de quaternions.* Lecture Notes in Mathematics, 800. Springer, Berlin (1980).

[Vo] Voisin, C. *Hodge theory and complex algebraic geometry I.*, trans. Schneps, L. Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge (2002).

[Wa] Waldspurger, J.-L. *Correspondances de Shimura et quaternions.* Forum Math. 3 (1991), no. 3, 219-307.

[We] Weil, Andre. *Basic number theory.*Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, New York - Berlin (1967).

[YZZ] Yuan, X., Zhang, S., and Zhang, W. *The Gross-Zagier Formula on Shimura Curves.* Ann. of Math. Studies, Princeton University Press, Princeton. (2013).

[Zh1] Zhang, Shouwu, *Heights of Heegner cycles and derivatives of L-series*, Invent. Math. 130 (1997), no. 1, 99-152.

[Zh2] Zhang, Shouwu, *Heights of Heegner points on Shimura Curves*, Ann. of Math. (2) 153 (2001), 27-147.