Topics in Stochastic Control with Applications to Finance

by

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To my mom, dad, and sister

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The most fascinating, and perhaps puzzling, part of being a teacher is that while you may be able to shape the life of a student, you never know exactly how and when, and even whether, it happens—unless, a student, after many years, come up to you with a big thank you, and tells you his story.

I am now telling my story.

Throughout the first twenty-two years of my life, I did not have a thought of being a mathematician. I went to the top business school in my home country, as a Finance major. Classes there were of the standard "business-school" style: a professor always told jokes, and tried his best to engage students by creating spirited discussions. Professor Chyi-Mei Chen stood out, by doing none of those. His class was composed of mathematical formulations and reasoning and proofs. We suffered. But through the suffering I learned two things. First, I suffered much less than others. Second, I felt that my mind had been opened to a new language, a language that could reasonably describe and explain financial markets. I then started to take courses in mathematics, discovered with surprise the joy of doing math, and eventually took Mathematics as another major. In my last year of college, I decided to pursue a Ph.D. degree in Mathematics. I would like to thank Professor Chiy-Mei Chen for his profound influence on me, though he did not have any clue of it.

I applied to ten Ph.D. programs in the US. I really appreciate the effort of Professor Tai-Chia Lin at National Taiwan University and the insight of Professor Sijue

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At Michigan, I took courses, attended seminars, but there was always some anxiousness inside: "I chose to do math because I like it, but am I really capable enough of doing research in math?" Then, I met Erhan, the one who trained me in various aspects. Erhan is a true mathematician, but to me, he is more like an extremely energetic businessman. A normal businessman looks for profits, and Erhan looks for interesting math problems. He is very successful in the sense that he can always find tons of very interesting and meaningful math problems. Moreover, he is so energetic that he can work on those problems at the same time. I am deeply grateful that he took me along with him for the past four years, giving me the chance to assist in his business. This experience led me to believe that I made the right choice to do math. From now on, I will start to have my own business, and "Erhan's business model" will always be on my mind.

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So, this is my story. Fueled with a full tank of gratitude, I am ready to write another new chapter.

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ABSTRACT

Topics in Stochastic Control with Applications to Finance

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This thesis is devoted to PDE characterization for stochastic control problems when the classical methodology of dynamic programming does not work. Under the framework of viscosity solutions, a dynamic programming principle serves as the tool to associate a (nonlinear) PDE to a stochastic control problem. Furthermore, if the associated PDE enjoys a comparison principle, then the stochastic control problem is fully characterized as the unique viscosity solution to the PDE. Unfortunately, a dynamic programming principle is in general difficult to prove, and may fail to be true in some cases. In Chapters II, III, and IV, we investigate different scenarios where classical dynamic programming does not work, and propose various methods to circumvent this obstacle.

In Chapter II, following the Stochastic Portfolio Theory, we consider quantile hedging in a market which admits arbitrage. A classical dynamic programming principle does not hold as this market allows for non-Lipschitz coefficients with super-linear growth in the state dynamics. By employing a mixture of convex duality, elliptic regularization, and stability of viscosity solutions, we characterize the quantile hedging problem as the smallest nonnegative viscosity supersolution to a fully

nonlinear PDE.

In Chapter III, we study robust growth-optimal trading: how to maximize the growth rate of one's wealth in a robust manner, when precise dynamics of the underlying assets is not known? This problem falls under the umbrella of ergodic control, for which the dynamic programming heuristic cannot be directly applied. By resorting to the spectral theory for fully nonlinear elliptic operators, we identify a robust trading strategy in terms of the principal eigenvalue of a fully nonlinear elliptic operator and its associated eigenfunction.

In Chapter IV, we investigate a zero-sum stochastic differential game of control and stopping. While the stopper intends to minimize her expected cost by choosing an optimal stopping strategy (a function mapping a control to a stopping time), the controller manipulates the state process to frustrate the stopper's effort. Since there is no measurable selection result for stopping strategies, classical dynamic programming does not work. We instead formulate two suitable weak dynamic programming principles, and use them to characterize the value of this game as the unique viscosity solution to an obstacle problem of a Hamilton-Jacobi-Bellman equation.

CHAPTER I

Introduction

Stochastic control theory is known to be an essential building block of mathematical finance. Typically, we model the evolution of asset prices by some stochastic process. Our wealth can then be formulated as another stochastic process, which can be controlled by choosing different trading strategies. In most applications, our goal is to optimize the expected value of some functional of the asset price process or the wealth process, by choosing a suitable control (trading strategy). Following the standard approach of dynamic programming originated in the theory of deterministic control, we may derive the corresponding dynamic programming principle under our stochastic context. With the aid of the theory of viscosity solutions developed by Crandall, Ishii, & Lions [30], the above dynamic programming principle implies that the optimal expected value we consider can be characterized as the viscosity solution to some (fully nonlinear) partial differential equation (PDE). While the above methodology has now been well-understood (with detailed and readable accounts in Fleming & Soner [48], Pham [91], and Touzi [107], among others), it is not always applicable. The main limitation is that deriving a standard dynamic programming principle requires **(i)** a priori regularity of the value function, and **(ii)** the technique of measurable selection. In general, the regularity required may not be true, and

measurable selection could be very difficult to implement.

In each of the three chapters, Chapters II-IV, of this thesis, we intend to obtain a PDE characterization for some optimal expected value, but are faced with the situation where a standard dynamic programming principle does not hold, due to some degeneracy inherent in the market we consider. To overcome this, we either develop a weaker form of a standard dynamic programming principle, or construct new tools and methods to tackle the problem from a different perspective.

1.1   Outline of this thesis

Chapter II studies the quantile hedging problem in the presence of arbitrage opportunity. The Stochastic Portfolio Theory (SPT), which began in Fernholz [44], attempts to *describe* observable phenomena in equity markets, including the presence of arbitrage. Following the SPT, we assume only the existence of a local martingale deflator (instead of an equivalent local martingale measure) such that the market may admit arbitrage. Moreover, we assume only the existence of a weak solution to the state dynamics, which allows non-Lipschitz coefficients with super-linear growth. Our goal is to characterize the smallest initial capital needed for quantile hedging.

First, we notice that the general PDE characterization for quantile hedging problems introduced in Bouchard, Elie & Touzi [21] cannot be applied here, as their main tool, the geometric dynamic programming principle, relies on the existence of a unique strong solution. Instead of performing dynamic programming, we employ a mixture of convex duality and elliptic regularization to characterize the quantile hedging problem in our case as the smallest nonnegative viscosity supersolution to a fully nonlinear PDE.

This chapter is based on Bayraktar, Huang & Song [9]. Parts of this work have

been presented in the 2010 World Congress of the Bachelier Finance Society (June 23, 2010), the Financial/Actuarial Mathematics Seminar at the University of Michigan (September 16, 2010), Workshop on Stochastic Analysis in Finance and Insurance at the University of Michigan (May 18, 2011).

Chapter III considers the problem of robust growth-optimal trading: how to maximize the growth rate of one's wealth in a robust manner, when precise dynamics of the underlying assets is not known? If the uncertainty lies only in the drift coefficient of the underlying assets, this problem was covered in Kardaras & Robertson [73]. We intend to study the case where *even* the covariance structure is not known precisely. Our goal is to determine a robust trading strategy under which the corresponding wealth process always attains the robust maximal asymptotic growth rate, no matter which admissible covariance structure materializes.

First, we observe that the associated differential operator under covariance uncertainty, denoted by F, is a variant of Pucci's extremal operator. We define the "principal eigenvalue" of F, denoted by λ^*, in some appropriate sense. Thanks to the spectral theory for fully nonlinear elliptic operators, we obtain a Harnack inequality for F. By using the Harnack inequality, the stability of viscosity solutions, and a regularity result for fully nonlinear elliptic operators in Safonov [101], we establish the relation $\lambda^* = \inf_c \lambda^{*,c}$, where $\lambda^{*,c}$ is the principal eigenvalue in [73] (with the covariance structure c a priori given) and the infimum is taken over all the admissible covariance structures. This implies that we can approximate the current problem with covariance uncertainty by a sequence of problems in [73] with fixed covariances. This approximation then enables us to characterize the robust maximal asymptotic growth rate as λ^*, and identify a robust trading strategy in terms of λ^* and its corresponding eigenfunction.

This chapter is based on Bayraktar & Huang [8]. Parts of this work have been presented in Probability, Control and Finance, a conference in honor of Ioannis Karatzas at Columbia University (June 5, 2012), the 2012 SIAM Conference on Financial Mathematics and Engineering (July 9, 2012), Department of Mathematics at Imperial College London (February 6, 2013), Department of Mathematics and Statistics at McMaster University (February 13, 2013), and School of Mathematical Sciences at Dublin City University (February 18, 2013).

Chapter IV investigates a zero-sum stochastic differential game of control and stopping. The controller affects the state process X^α by selecting the control α; on the other hand, the stopper decides the duration of this game, but incurs both running and terminal cost. While the stopper intends to stop optimally so that her expected discounted cost can be minimized, the controller manipulates X^α to frustrate the effort of the stopper. Although this type of game has been studied under several different methods (see e.g. Karatzas & Zamfirescu [71], Hamadène & Lepeltier [55], and Hamadène [54]), all of them require the diffusion coefficient of X^α be *non-degenerate* and *control-independent*. Without imposing these restrictions, we intend to determine under what conditions the game has a value, and derive a PDE characterization for this value when it exists.

Our method is motivated by Bouchard & Touzi [24], where the weak dynamic programming principle (WDPP) was introduced. First, we give appropriate definitions of the upper (resp. lower) value function U (resp. V) for the controller-stopper game. By generalizing [24] to current context, we derive two different WDPPs: one for V and one for U^* (here, U^* is the upper semicontinuous envelope of U). The WDPP for V implies that V is a viscosity supersolution to an obstacle problem for a Hamilton-Jacobi-Bellman equation; similarly, the WDPP for U^* gives the viscosity

subsolution property of U^* to the same obstacle problem. Next, by proving a comparison theorem for this obstacle problem, we obtain $U^* \leq V$. Recalling that $U \geq V$ by definition, we conclude that $U^* = V$. This in particular implies $U = V$, i.e. the game has a value, and the value function is characterized as the unique viscosity solution to the associated obstacle problem.

This chapter is based on Bayraktar & Huang [7]. Parts of this work have been presented in the 2010 Mathematical Finance and Partial Differential Equations Conference at Rutgers University (December 10, 2010), the 2011 International Congress on Industrial and Applied Mathematics (July 21, 2011), the Financial/Actuarial Mathematics Seminar at the University of Michigan (September 29, 2011), and the 2012 SIAM Conference on Financial Mathematics and Engineering (July 10, 2012).

CHAPTER II

Outperforming the Market Portfolio with a Given Probability

2.1 Introduction

In this chapter we consider the quantile hedging problem when the underlying market may not have an equivalent martingale measure. Instead, we assume that there exists a *local martingale deflator* (a strict local martingale which when multiplied by the asset prices yields a positive local martingale). We characterize the value function as the smallest nonnegative viscosity supersolution of a fully non-linear partial differential equation (PDE). This resolves an open problem proposed in the final section of [40]; also see pages 61 and 62 of [99].

Our framework falls under the umbrella of the Stochastic Portfolio Theory of Fernholz and Karatzas, see e.g. [45], [47], [46]; and the benchmark approach of Platen [93]. Under this framework, the linear PDE satisfied by the superhedging price does not have a unique solution; see e.g. [41], [46], [42], and [100]. Similar phenomena occur when the asset prices have *bubbles*: an equivalent local martingale measure exists, but the asset prices under this measure are strict local martingales; see e.g. [29], [58], [60], [61], [35], and [13]. A related series of papers [1], [102], [79], [59], [78], [36], and [12] addressed the issue of bubbles in the context of stochastic volatility models. In particular, [12] gave necessary and sufficient conditions for linear PDEs

appearing in the context of stochastic volatility models to have a unique solution.

In contrast, we show that the quantile hedging problem, which is equivalent to a stochastic control problem, is a viscosity supersolution to a fully non-linear PDE. As in the linear case, this PDE may not have a unique solution. Therefore, a further characterization for the value function is needed. Our main result shows that the value function is not only a viscosity supersolution, but the smallest nonnegative one, to the associated fully nonlinear PDE. Recently, [63], [11], and [43] also considered stochastic control problems in this framework. The first reference solves the classical utility maximization problem, the second one solves the optimal stopping problem, whereas the third one determines the optimal arbitrage under model uncertainty, which is equivalent to solving a zero-sum stochastic game.

The structure of the chapter is simple: In Section 2.2, we formulate the problem. In this section we also discuss the implications of assuming the existence of a local martingale deflator. In Section 2.3, we generalize the results of [50] on quantile hedging, in particular the Neyman-Pearson Lemma. We also prove other properties of the value function such as convexity. Section 2.4 is where we give the PDE characterization of the value function.

2.2 The model

We consider a financial market with a bond which is always equal to 1, and d stocks $X = (X_1, \cdots, X_d)$ which satisfy

$$(2.2.1) \qquad dX_i(t) = X_i(t) \left(b_i(X(t))dt + \sum_{k=1}^{d} s_{ik}(X(t))dW_k(t) \right), i = 1, \cdots d,$$

with the initial condition $X(0) = x = (x_1, \cdots, x_d) \in (0, \infty)^d$. Here, $W(\cdot) := (W_1(\cdot), \cdots, W_d(\cdot))$ is a d-dimensional Brownian motion.

Following the set up in [41, Section 8], we make the following assumption.

Assumption II.1. *Let $b_i : (0, \infty)^d \to \mathbb{R}$ and $s_{ik} : (0, \infty)^d \to \mathbb{R}$ be continuous functions. Set $b(\cdot) = (b_1(\cdot), \cdots, b_d(\cdot))'$ and $s(\cdot) = (s_{ij}(\cdot))_{1 \le i,j \le d}$, which we assume to be invertible for all $x \in (0, \infty)^d$. We also assume that (2.2.1) has a weak solution that is unique in distribution for every initial value. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space specified by a weak solution. Another assumption we will impose is that*

$$(2.2.2) \qquad \sum_{i=1}^{d} \int_0^T \left(|b_i(X(t))| + a_{ii}(X(t)) + \theta_i^2(X(t)) \right) dt < \infty, \ \mathbb{P}\text{-}a.s,$$

where $\theta(\cdot) := s^{-1}(\cdot)b(\cdot)$, $a_{ij}(\cdot) := \sum_{k=1}^{d} s_{ik}(\cdot)s_{jk}(\cdot)$.

We will denote by $\mathbb{F} = \{\mathcal{F}_t\}_{t \ge 0}$ the right-continuous version of the natural filtration generated by $X(\cdot)$, and by $\mathbb{G}$ the $\mathbb{P}$-augmentation of the filtration $\mathbb{F}$. Thanks to Assumption II.1, the Brownian motion $W(\cdot)$ of (2.2.1) is adapted to $\mathbb{G}$ (see e.g. [41, Section 2]), every local martingale of $\mathbb{F}$ has the martingale representation property, i.e. it can be represented as a stochastic integral, with respect to $W(\cdot)$, of some $\mathbb{G}$-progressively measurable integrand (see e.g. the discussion on p.1185 in [41]), the solution of (2.2.1) takes values in the positive orthant, and the exponential local martingale

$$(2.2.3) \quad Z(t) := \exp\left\{ -\int_0^t \theta(X(s))'dW(s) - \frac{1}{2}\int_0^t |\theta(X(s))|^2 ds \right\}, \quad 0 \le t < \infty,$$

the so-called *deflator* is well defined. We do not exclude the possibility that $Z(\cdot)$ is a strict local martingale.

Let $\mathcal{H}$ be the set of $\mathbb{G}$-progressively measurable processes $\pi : [0, T) \times \Omega \to \mathbb{R}^d$, which satisfies

$$\int_0^T \left(|\pi(t)'\mu(X(t))| + \pi(t)'\alpha(X(t))\pi(t) \right) dt < \infty, \quad \mathbb{P}\text{-a.s.},$$

in which $\mu = (\mu_1, \cdots, \mu_d)$ and $\sigma = (\sigma_{ij})_{1 \le i,j \le d}$ with $\mu_i(x) = b_i(x)x_i$, $\sigma_{ik}(x) = s_{ik}(x)x_i$, and $\alpha(x) = \sigma(x)\sigma(x)'$.

At time t, an investor invests $\pi_i(t)$ proportion of his wealth in the ith stock. The proportion $1 - \sum_{i=1}^{d} \pi_i(t)$ gets invested in the bond. For each $\pi \in \mathcal{H}$ and initial wealth $y \geq 0$ the associated wealth process will be denoted by $Y^{y,\pi}(\cdot)$. This process solves

$$dY^{y,\pi}(t) = Y^{y,\pi}(t) \sum_{i=1}^{d} \pi_i(t) \frac{dX_i(t)}{X_i(t)}, \quad Y^{y,\pi}(0) = y.$$

It can be easily seen that $Z(\cdot)Y^{y,\pi}(\cdot)$ is a positive local martingale for any $\pi \in \mathcal{H}$. Let $g : (0,\infty)^d \to (0,\infty)$ be a measurable function satisfying

(2.2.4) $$\mathbb{E}[Z(T)g(X(T))] < \infty,$$

and define

$$V(T,x,1) := \inf\{y > 0 : \exists \pi(\cdot) \in \mathcal{H} \text{ s.t. } Y^{y,\pi}(T) \geq g(X(T))\}.$$

Thanks to Assumption II.1, we have that $V(T,x,1) = \mathbb{E}[Z(T)g(X(T))]$; see e.g. [46, Section 10]. Note that if g has linear growth, then (2.2.4) is satisfied since the process ZX is a positive supermartingale.

2.2.1 A Digression: What does the existence of a local martingale deflator entail?

Although, we do not assume the existence of equivalent local martingale measures, we assume the existence of a local martingale deflator. This is equivalent to the *No-Unbounded-Profit-with-Bounded-Risk* (NUPBR) condition; see [63, Theorem 4.12]. NUPBR is defined as follows: A sequence (π^n) of admissible portfolios is said to generate a UPBR if $\lim_{m \to \infty} \sup_n \mathbb{P}[Y^{1,\pi^n}(T) > m] > 0$. If no such sequence exists, then we say that NUPBR holds; see [63, Proposition 4.2]. In fact, the so-called *No-Free-Lunch-with-Vanishing-Risk* (NFLVR) is equivalent to NUPBR plus the classical *no-arbitrage* assumption. Thus, in our setting (since we assumed the existence of local martingale deflators), although arbitrages exist they remain on the level of

"cheap thrills", which was coined by [80]. (Note that the results of Karatzas and Kardaras [63] also imply that one does not need NFLVR for the portfolio optimization problem of an individual to be well-defined. One merely needs the NUPBR condition to hold.) The failure of no-arbitrage means that the money market is not an optimal investment and is dominated by other investments. It follows that a short position in the money market and long position in the dominating assets leads one to arbitrage. However, one can not scale the arbitrage and make an arbitrary profit because of the admissibility constraint, which requires the wealth to be positive. This is what is contained in NUPBR, which holds in our setting. Also, see [72], where these issues are further discussed.

2.3 On quantile hedging

In this section, we develop new probabilistic tools to extend results of Föllmer and Leukert [50] on quantile hedging to settings where equivalent martingale measures need not exist. This is not only mathematically intriguing, but also economically important because it admits arbitrage in the market, which opens the door to the notion of optimal arbitrage, recently introduced in Fernholz and Karatzas [41]. The tools in this section facilitate the discussion of quantile hedging under the context of optimal arbitrage, leading us to generalize the results of [41] on this sort of probability-one outperformance.

We intend to determine

$$(2.3.1) \qquad V(T, x, p) = \inf\{y > 0 \,|\, \exists\, \pi \in \mathcal{H} \text{ s.t. } \mathbb{P}\{Y^{y,\pi}(T) \geq g(X(T))\} \geq p\},$$

for $p \in [0, 1]$. Note that the set on which we take infimum in (2.3.1) is nonempty. Indeed, under the condition (2.2.4), there exists $\pi \in \mathcal{H}$ such that $Y^{y,\pi}(T) = g(X(T))$ a.s., where $y := \mathbb{E}[Z(T)g(X(T))]$; see e.g. [46, Section 10]. It follows that for any

$p \in [0, 1]$,

$$\mathbb{P}\{Y^{y,\pi}(T) \geq g(X(T))\} = 1 \geq p.$$

Also observe that

$$\widetilde{V}(T, x, p) := \frac{V(T, x, p)}{g(x)} = \inf\{r > 0|\, \exists\, \pi \in \mathcal{H} \text{ s.t. } \mathbb{P}\{Y^{rg(x),\pi}(T) \geq g(X(T))\} \geq p\}.$$

When $g(x) = \sum_{i=1}^{d} x_i$, observe that $\widetilde{V}(T, x, 1)$ is equal to equation (6.1) of [41], the smallest relative amount to beat the market capitalization $\sum_{i=1}^{d} X_i(T)$.

Remark 2.3.1. Clearly,

$$(2.3.2) \qquad 0 = V(T, x, 0) \leq V(T, x, p) \nearrow V(T, x, 1) \leq g(x), \quad \text{as } p \to 1.$$

By analogy with [50], we shall present a probabilistic characterization of $V(T, x, p)$. First, we will generalize the Neyman-Pearson lemma (see e.g. [51, Theorem A.28]) in the next result.

Lemma 2.3.2. *Suppose Assumption II.1 holds and g satisfies (2.2.4). Let $A \in \mathcal{F}_T$ satisfy*

$$(2.3.3) \qquad\qquad\qquad \mathbb{P}(A) \geq p.$$

Then

$$(2.3.4) \qquad\qquad\qquad V(T, x, p) \leq \mathbb{E}[Z(T)g(X(T))\mathbf{1}_A].$$

Furthermore, if $A \in \mathcal{F}_T$ satisfies (2.3.3) with equality and

$$(2.3.5) \qquad \operatorname{ess\,sup}_A\{Z(T)g(X(T))\} \leq \operatorname{ess\,inf}_{A^c}\{Z(T)g(X(T))\},$$

then A satisfies (2.3.4) with equality.

Proof. Under Assumption II.1, since $g(X(T))1_A \in \mathcal{F}_T$ satisfies condition (2.2.4), it is replicable with initial capital $y := \mathbb{E}[Z(T)g(X(T))1_A]$; see e.g. Section 10.1 of [46]. That is, there exists $\pi \in \mathcal{H}$ such that $Y^{y,\pi}(T) = g(X(T))1_A$ a.s. Now if $\mathbb{P}(A) \geq p$, we have $\mathbb{P}\{Y^{y,\pi}(T) \geq g(X(T))\} = \mathbb{P}\{1_A \geq 1\} \geq p$. Then it follows from (2.3.1) that $V(T, x, p) \leq y = \mathbb{E}[Z(T)g(X(T))1_A]$.

Now, take an arbitrary pair (y_0, π_0) of initial capital and admissible portfolio that replicates $g(X(T))$ with probability greater than or equal to p, i.e.

$$\mathbb{P}\{B\} \geq p, \text{ where } B := \{Y^{y_0,\pi_0}(T) \geq g(X(T))\}.$$

Let $A \in \mathcal{F}_T$ satisfy $p = \mathbb{P}(A) \leq \mathbb{P}(B)$ and (2.3.5). To prove equality in (2.3.4), it is enough to show that

$$y_0 \geq \mathbb{E}[Z(T)g(X(T))1_A],$$

which can be shown as follows:

$$\begin{aligned}
y_0 &\geq \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)] = \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)1_B] + \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)1_{B^c}] \\
&\geq \mathbb{E}[Z(T)g(X(T))1_B] = \mathbb{E}[Z(T)g(X(T))1_{A\cap B}] + \mathbb{E}[Z(T)g(X(T))1_{A^c\cap B}] \\
&\geq \mathbb{E}[Z(T)g(X(T))1_{A\cap B}] + \mathbb{P}(A^c \cap B) \operatorname{ess\,inf}_{A^c\cap B}\{Z(T)g(X(T))\} \\
&\geq \mathbb{E}[Z(T)g(X(T))1_{A\cap B}] + \mathbb{P}(A \cap B^c) \operatorname{ess\,sup}_{A\cap B^c}\{Z(T)g(X(T))\} \\
&\geq \mathbb{E}[Z(T)g(X(T))1_{A\cap B}] + \mathbb{E}[Z(T)g(X(T))1_{A\cap B^c}] \\
&= \mathbb{E}[Z(T)g(X(T))1_A],
\end{aligned}$$

where in the fourth inequality we use the following two observations: First, $\mathbb{P}(A^c \cap B) = \mathbb{P}(A \cup B) - \mathbb{P}(A) \geq \mathbb{P}(A \cup B) - \mathbb{P}(B) = \mathbb{P}(B^c \cap A)$. Second,

$$\begin{aligned}
\operatorname{ess\,inf}_{A^c\cap B}\{Z(T)g(X(T))\} &\geq \operatorname{ess\,inf}_{A^c}\{Z(T)g(X(T))\} \\
&\geq \operatorname{ess\,sup}_{A}\{Z(T)g(X(T))\} \\
&\geq \operatorname{ess\,sup}_{A\cap B^c}\{Z(T)g(X(T))\},
\end{aligned}$$

in which the second inequality follows from (2.3.5). $\qquad\square$

Let $F(\cdot)$ be the cumulative distribution function of $Z(T)g(X(T))$ and for any $a \in \mathbb{R}_+$ define

$$A_a := \{\omega : Z(T)g(X(T)) < a\}, \ \partial A_a := \{\omega : Z(T)g(X(T)) = a\},$$

and let $\bar{A}_a$ denote $A_a \cup \partial A_a$; that is,

$$(2.3.6) \qquad \bar{A}_a = \{\omega : Z(T)g(X(T)) \leq a\}.$$

Taking $A = \bar{A}_a$ in Lemma 2.3.2, we see that (2.3.5) is satisfied. It follows that

$$(2.3.7) \qquad V(T, x, F(a)) = \mathbb{E}[Z(T)g(X(T))1_{\bar{A}_a}].$$

On the other hand, taking $A = A_a$, we see that (2.3.5) is again satisfied. We therefore obtain

$$(2.3.8) \qquad V(T, x, F(a-)) = \mathbb{E}[Z(T)g(X(T))1_{A_a}].$$

The last two equalities imply the following relationship

$$(2.3.9) \qquad \begin{aligned} V(T, x, F(a)) \ &= V(T, x, F(a-)) + a\mathbb{P}\{\partial A_a\} \\ &= V(T, x, F(a-)) + a(F(a) - F(a-)). \end{aligned}$$

Next, we will determine $V(T, x, p)$ for $p \in (F(a-), F(a))$ when $F(a-) < F(a)$.

Proposition 2.3.3. *Suppose Assumption II.1 holds. Fix an* $(x, p) \in (0, \infty)^d \times [0, 1]$.

(i) *There exists* $A \in \mathcal{F}_T$ *satisfying* (2.3.3) *with equality and* (2.3.5). *As a result,* (2.3.4) *holds with equality.*

(ii) *If* $F^{-1}(p) := \{s \in \mathbb{R}_+ : F(s) = p\} = \emptyset$, *then letting* $a := \inf\{s \in \mathbb{R}_+ : F(s) > p\}$ *we have*

$$(2.3.10) \qquad \begin{aligned} V(T, x, p) \ &= V(T, x, F(a-)) + a(p - F(a-)). \\ &= V(T, x, F(a)) - a(F(a) - p) \end{aligned}$$

Proof. **(i)** If there exists $a \in \mathbb{R}$ such that either $F(a) = p$ or $F(a-) = p$, then we can take $A = A_a$ or $A = \bar{A}_a$, thanks to (2.3.7) and (2.3.8). In the rest of the proof we will assume that $F^{-1}(p) = \emptyset$.

Let $\widetilde{W}$ be a Brownian motion with respect to $\mathbb{F}$ and define $B_b = \{\omega : \frac{\widetilde{W}(T)}{\sqrt{T}} < b\}$. Let us define $f(\cdot)$ by $f(b) = \mathbb{P}\{\partial A_a \cap B_b\}$. The function f satisfies $\lim_{b \to -\infty} f(b) = 0$ and $\lim_{b \to \infty} f(b) = \mathbb{P}(\partial A_a)$. Moreover, the function $f(\cdot)$ is continuous and nondecreasing. Right continuity can be shown as follows: For $\varepsilon > 0$

$$0 \leq f(b+\varepsilon) - f(b) = \mathbb{P}(\partial A_a \cap B_{b+\varepsilon}) - \mathbb{P}(\partial A_a \cap B_b) \leq \mathbb{P}(B_{b+\varepsilon} \cap B_b^c).$$

The right continuity follows from observing that the last expression goes to zero as $\varepsilon \to 0$. One can show left continuity of $f(\cdot)$ in a similar fashion.

Since $0 < p - \mathbb{P}(A_a) < \mathbb{P}(\partial A_a)$, thanks to the above properties of f there exists $b^* \in \mathbb{R}$ satisfying $f(b^*) = p - \mathbb{P}(A_a)$.

Define $A := A_a \cup (\partial A_a \cap B_{b^*})$. Observe that $\mathbb{P}(A) = \mathbb{P}(A_a) + \mathbb{P}(\partial A_a \cap B_{b^*}) = p$ and that A satisfies (2.3.5).

(ii) This follows immediately from (i):

$$
\begin{aligned}
V(T, x, p) &= \mathbb{E}[Z(T)g(X(T))1_A] \\
&= \mathbb{E}[Z(T)g(X(T))1_{A_a}] + \mathbb{E}[Z(T)g(X(T))1_{\partial A_a \cap B_{b^*}}] \\
&= V(T, x, F(a-)) + a\mathbb{P}(\partial A_a \cap B_{b^*}) \\
&= V(t, x, F(a-)) + a(p - F(a-)).
\end{aligned}
$$

$\square$

Remark 2.3.4. Note that when Z is a martingale, using the Neyman-Pearson Lemma, it was shown in [50] that

$$(2.3.11) \qquad V(T, x, p) = \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi] = \mathbb{E}[Z(T)g(X(T))\varphi^*],$$

where

$$(2.3.12) \qquad \mathcal{M} = \left\{ \varphi : \Omega \to [0,1] \middle| \mathcal{F}_T \text{ measurable}, \mathbb{E}[\varphi] \geq p \right\}.$$

The randomized test function φ^* is not necessarily an indicator function. Using Lemma 2.3.2 and the fine structure of the filtration $\mathcal{F}_T$, we provide in Proposition 2.3.3 another optimizer of (2.3.11) which is an indicator function.

Proposition 2.3.5. *Suppose Assumption II.1 holds. Then, the map $p \mapsto V(T, x, p)$ is convex and continuous on the closed interval $[0,1]$. Hence, $V(T, x, p) \leq pV(T, x, 1) \leq pg(x)$ for all $p \in [0,1]$.*

Proof. By Proposition 2.3.3, for any $p \in [0,1]$ there exists $A \in \mathcal{F}_T$ such that

$$V(T, x, p) = \mathbb{E}[Z(T)g(X(T))1_A] \leq \mathbb{E}[Z(T)g(X(T))] < \infty.$$

Then thanks to a theorem by Ostroski (see [33, p.12]), to show the convexity it suffices to demonstrate the midpoint convexity

$$(2.3.13) \qquad \frac{V(T, x, p_1) + V(T, x, p_2)}{2} \geq V\left(T, x, \frac{p_1 + p_2}{2} \right), \quad \text{for all } 0 \leq p_1 < p_2 \leq 1.$$

Denote $\tilde{p} := \frac{p_1 + p_2}{2}$. It follows from Proposition 2.3.3 that there exist $A_1 \subset \tilde{A} \subset A_2$ with $\mathbb{P}(A_1) = p_1 < \mathbb{P}(\tilde{A}) = \tilde{p} < \mathbb{P}(A_2) = p_2$ satisfying (2.3.5),

$$V(T, x, p_i) = \mathbb{E}[Z(T)g(X(T))1_{A_i}], \ i = 1, 2,$$

and

$$V(T, x, \tilde{p}) = \mathbb{E}[Z(T)g(X(T))1_{\tilde{A}}].$$

By (2.3.5),

$$\begin{aligned}
\operatorname{ess\,inf}\{Z(T)g(X(T))1_{A_2 \cap \tilde{A}^c}\} &\geq \operatorname{ess\,inf}\{Z(T)g(X(T))1_{\tilde{A}^c}\} \\
&\geq \operatorname{ess\,sup}\{Z(T)g(X(T))1_{\tilde{A}}\} \\
&\geq \operatorname{ess\,sup}\{Z(T)g(X(T))1_{\tilde{A} \cap A_1^c}\},
\end{aligned}$$

which implies that $\mathbb{E}[Z(T)g(X(T))1_{A_2 \cap \tilde{A}^c}] \geq \mathbb{E}[Z(T)g(X(T))1_{\tilde{A} \cap A_1^c}]$. As a result,

$$\mathbb{E}[Z(T)g(X(T))1_{A_2}] - \mathbb{E}[Z(T)g(X(T))1_{\tilde{A}}]$$

$$\geq \mathbb{E}[Z(T)g(X(T))1_{\tilde{A}}] - \mathbb{E}[Z(T)g(X(T))1_{A_1}],$$

which is equivalent to (2.3.13).

Now thanks to convexity, we immediately have that $p \mapsto V(T, x, p)$ is continuous on $[0, 1)$. It remains to show that it is continuous from the left at $p = 1$; but this is indeed true because

$$
\begin{aligned}
\lim_{a \to \infty} V(T, x, F(a)) &= \lim_{a \to \infty} \mathbb{E}[Z(T)g(X(T))1_{\{Z(T)g(X(T)) \leq a\}}] \\
&= \mathbb{E}[Z(T)g(X(T))] = V(T, x, 1),
\end{aligned}
$$

where the second equality is due to the dominated convergence theorem. $\qquad \square$

Example 2.3.6. Consider a market with a single stock, whose dynamics follow a three-dimensional Bessel process, i.e.

$$dX(t) = \frac{1}{X(t)} dt + dW(t) \quad X_0 = x > 0,$$

and let $g(x) = x$. In this case $Z(t) = x/X(t)$, which is the classical example for a strict local martingale; see [62]. On the other hand, $Z(t)X(t) = x$ is a martingale. Thanks to Proposition 2.3.3 there exists a set $A \in \mathcal{F}_T$ with $\mathbb{P}(A) = p$ such that

$$V(T, x, p) = \mathbb{E}[Z(T)X(T)1_A] = px.$$

In [50], the following result was proved when Z is a martingale. Here, we generalize this result to the case where Z is only a local martingale.

Proposition 2.3.7. *Under Assumption II.1*

(2.3.14) $$V(T, x, p) = \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi],$$

where $\mathcal{M}$ is defined in (2.3.12).

Proof. Thanks to Proposition 2.3.3 there exists a set $A \in \mathcal{F}_T$ satisfying $\mathbb{P}(A) = p$ and (2.3.5) such that $V(T, x, p) = \mathbb{E}[Z(T)g(X(T))1_A]$. Since $1_A \in \mathcal{M}$, clearly

$$V(T, x, p) \geq \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi].$$

For the other direction, it is enough to show that for any $\varphi \in \mathcal{M}$, we have

$$\mathbb{E}[Z(T)g(X(T))1_A] \leq \mathbb{E}[Z(T)g(X(T))\varphi].$$

Indeed, since the left hand side is actually $V(T, x, p)$, we get the desired result by taking infimum on both sides over $\varphi \in \mathcal{M}$. Now, taking $M := \operatorname{ess\,sup}_A\{Z(T)g(X(T))\}$, we observe that

$$
\begin{aligned}
&\mathbb{E}[Z(T)g(X(T))\varphi] - \mathbb{E}[Z(T)g(X(T))1_A] \\
&= \mathbb{E}[Z(T)g(X(T))\varphi 1_A] + \mathbb{E}[Z(T)g(X(T))\varphi 1_{A^c}] - \mathbb{E}[Z(T)g(X(T))1_A] \\
&= \mathbb{E}[Z(T)g(X(T))\varphi 1_{A^c}] - \mathbb{E}[Z(T)g(X(T))1_A(1 - \varphi)] \\
&\geq \operatorname{ess\,inf}_{A^c}\{Z(T)g(X(T))\}\mathbb{E}[\varphi 1_{A^c}] - M\mathbb{E}[1_A(1 - \varphi)] \\
&\geq M\mathbb{E}[\varphi 1_{A^c}] - M\mathbb{E}[1_A(1 - \varphi)] \quad \text{(by (2.3.5))} \\
&= M\mathbb{E}[\varphi] - M\mathbb{E}[1_A] \geq 0.
\end{aligned}
$$

$\square$

2.3.1 A Digression: Representation of V as a Stochastic Control Problem

For $p \in [0, 1]$, we introduce an additional controlled state variable

$$(2.3.15) \qquad P_\alpha^p(s) = p + \int_0^s \alpha(r)'dW(r), \ \ s \in [0, T],$$

where $\alpha(\cdot)$ is a $\mathbb{G}-$progressively measurable $\mathbb{R}^d$-valued process satisfying the integrability condition $\int_0^T |\alpha(s)|^2 ds < \infty$ a.s. such that P_α^p takes values in $[0, 1]$. We will denote the class of such processes by $\mathcal{A}$. Note that $\mathcal{A}$ is nonempty, as the constant control $\alpha(\cdot) \equiv (0, \cdots, 0) \in \mathbb{R}^d$ obviously lies in $\mathcal{A}$. The next result obtains an alternative representation for V in terms of P_α^p.

Proposition 2.3.8. *Under Assumption II.1,*

(2.3.16) $$V(T, x, p) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}[Z(T)g(X(T))P_\alpha^p(T)] < \infty.$$

Proof. The finiteness follows from (2.2.4). Define

$$\widetilde{\mathcal{M}} := \left\{ \varphi : \Omega \to [0, 1] \Big| \mathcal{F}_T \text{ measurable}, \mathbb{E}[\varphi] = p \right\}.$$

Thanks to Proposition 2.3.3, there exists a set $A \in \mathcal{F}_T$ satisfying $\mathbb{P}(A) = p$ and (2.3.5) such that

$$V(T, x, p) = \mathbb{E}[Z(T)g(X(T))1_A] \geq \inf_{\varphi \in \widetilde{\mathcal{M}}} \mathbb{E}[Z(T)g(X(T))\varphi].$$

Since the opposite inequality follows immediately from Proposition 2.3.7, we conclude

$$V(T, x, p) = \inf_{\varphi \in \widetilde{\mathcal{M}}} \mathbb{E}[Z(T)g(X(T))\varphi].$$

Therefore, it is enough to show that $\widetilde{\mathcal{M}}$ satisfies $\widetilde{\mathcal{M}} = \left\{ P_\alpha^p(T) \big| \alpha \in \mathcal{A} \right\}$. The inclusion $\widetilde{\mathcal{M}} \supset \left\{ P_\alpha^p(T) \big| \alpha \in \mathcal{A} \right\}$ is clear. To show the other inclusion we will use the Martingale representation theorem: For any $\varphi \in \widetilde{\mathcal{M}}$ there exists a $\mathbb{G}-$progressively measurable $\mathbb{R}^d$-valued process $\psi(\cdot)$ satisfying $\int_0^T |\psi(s)|^2 ds < \infty$ a.s. such that

$$\mathbb{E}[\varphi|\mathcal{F}_t] = p + \int_0^t \psi(s)'dW(s), \ t \in [0, T].$$

Note that since φ takes values in $[0, 1]$, so does $\mathbb{E}[\varphi|\mathcal{F}_t]$ for all $t \in [0, T]$. Then we see that $\mathbb{E}[\varphi|\mathcal{F}_t]$ satisfies (2.3.15) with $\alpha(\cdot) = \psi(\cdot) \in \mathcal{A}$. $\qquad\square$

2.4 The PDE characterization

2.4.1 Notation

We denote by $X^{t,x}(\cdot)$ the solution of (2.2.1) starting from x at time t, and by $Z^{t,x,z}(\cdot)$ the solution of

(2.4.1) $$dZ(s) = -Z(s)\theta(X^{t,x}(s))'dW(s), \ Z(t) = z.$$

Define the process $Q^{t,x,q}(\cdot)$ by

$$(2.4.2) \qquad Q^{t,x,q}(\cdot) := \frac{1}{Z^{t,x,(1/q)}(\cdot)}, \quad q \in (0,\infty).$$

Then we see from (2.4.1) that $Q(\cdot)$ satisfies

$$(2.4.3) \qquad \frac{dQ(s)}{Q(s)} = |\theta(X^{t,x}(s))|^2 ds + \theta(X^{t,x}(s))' dW(s), \ Q^{t,x,q}(t) = q.$$

We then introduce the value function

$$U(t,x,p) := \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z^{t,x,1}(T)g(X^{t,x}(T))\varphi],$$

where $\mathcal{M}$ is defined in (2.3.12). Note that the original value function V can be written in terms of U as $V(T,x,p) = U(0,x,p)$.

We also consider the Legendre transform of U with respect to the p variable. To make the discussion clear, let us first extend the domain of the map $p \mapsto U(t,x,p)$ from $[0,1]$ to the entire real line $\mathbb{R}$ by setting

$$(2.4.4) \qquad U(t,x,p) \;=\; 0 \text{ for } p < 0,$$

$$(2.4.5) \qquad U(t,x,p) \;=\; \infty \text{ for } p > 1.$$

Then the Legendre transform of U with respect to p is well-defined

$$(2.4.6) \ \ w(t,x,q) := \sup_{p \in \mathbb{R}} \{pq - U(t,x,p)\} = \begin{cases} \infty, & \text{if } q < 0; \\ \sup_{p \in [0,1]} \{pq - U(t,x,p)\}, & \text{if } q \geq 0. \end{cases}$$

From Proposition 2.3.5, we already know that $p \mapsto U(t,x,p)$ is convex and continuous on $[0,1]$. Since $U(t,x,0) = 0$, we see from (2.4.4) and (2.4.5) that $p \mapsto U(t,x,p)$ is continuous on $(-\infty, 1]$ and lower semicontinuous on $\mathbb{R}$. Moreover, considering that $p \mapsto U(t,x,p)$ is increasing on $[0,1]$, we conclude that $p \mapsto U(t,x,p)$ is also convex on $\mathbb{R}$. Now thanks to [108, §6.18], the convexity and the lower semicontinuity of

$p \mapsto U(t, x, p)$ on $\mathbb{R}$ imply that the double transform of U is indeed equal to U itself. That is, for any $(t, x, p) \in [0, T] \times (0, \infty)^d \times \mathbb{R}$,

$$U(t, x, p) = \sup_{q \in \mathbb{R}} \{pq - w(t, x, q)\} = \sup_{q \geq 0} \{pq - w(t, x, q)\},$$

where the second equality is a consequence of (2.4.6).

In this section, we also consider the function

(2.4.7)

$$\widetilde{w}(t, x, q) := \mathbb{E}[Z^{t,x,1}(T)(Q^{t,x,q}(T) - g(X^{t,x}(T)))^+] = \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))^+],$$

for any $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$. We will show that $w = \widetilde{w}$ and derive various properties of $\widetilde{w}$.

Remark 2.4.1. From the definition of $\widetilde{w}$ in (2.4.7), $\widetilde{w}$ is the upper hedging price for the contingent claim $(Q^{t,x,q}(T) - g(X^{t,x}(T)))^+$, and potentially solves the linear PDE

(2.4.8) $$\partial_t \widetilde{w} + \frac{1}{2} \text{Tr}(\sigma \sigma' D_x^2 \widetilde{w}) + \frac{1}{2}|\theta|^2 q^2 D_q^2 \widetilde{w} + q \text{Tr}(\sigma \theta D_{xq} \widetilde{w}) = 0.$$

This is not, however, a traditional Black-Scholes type equation because it is degenerate on the entire space $(x, q) \in (0, \infty)^d \times (0, \infty)$. Consider the following function v which takes values in the space of $(d+1) \times d$ matrices:

$$v(\cdot) := \left[\begin{array}{c} s(\cdot)_{d \times d} \\ \hline \theta(\cdot)'_{1 \times d} \end{array} \right]$$

Degeneracy can be seen by observing that $v(x)v(x)'$ is only positive semi-definite for all $x \in (0, \infty)^d$. Or, one may observe degeneracy by noting that there are $d+1$ risky assets, $X_1, \cdots, X_d$, and Q, with only d independent sources of uncertainty, $W_1, \cdots, W_d$. As a result, the existence of classical solutions to (2.4.8) cannot be

guaranteed by standard results for parabolic equations. Indeed, under the setting of Example 2.3.6, we have

$$\widetilde{w}(t, x, q) = \mathbb{E}[(q - Z^{t,x,1}(T)X^{t,x}(T))^+] = (q - x)^+,$$

which is not smooth.

2.4.2 Elliptic Regularization

In this subsection, we will approximate $\widetilde{w}$ by a sequence of smooth functions $\widetilde{w}_\varepsilon$, constructed by elliptic regularization. We will then derive some properties of $\widetilde{w}_\varepsilon$ and investigate the relation between $\widetilde{w}$ and $\widetilde{w}_\varepsilon$. Finally, we will show that $\widetilde{w} = w$, which validates the construction of $\widetilde{w}_\varepsilon$.

To perform elliptic regularization under our setting, we need to first introduce a product probability space. Recall that we have been working on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, given by a weak solution to the SDE (2.2.1). Now consider the sample space $\Omega^B := C([0, T]; \mathbb{R})$ and the canonical process $B(\cdot)$. Let $\mathbb{F}^B$ be the filtration generated by B and $\mathbb{P}^B$ be the Wiener measure on $(\Omega^B, \mathbb{F}^B)$. We then introduce the product probability space $(\bar{\Omega}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$, with $\bar{\Omega} := \Omega \times \Omega^B$, $\bar{\mathbb{F}} := \mathbb{F} \times \mathbb{F}^B$ and $\bar{\mathbb{P}} := \mathbb{P} \times \mathbb{P}^B$. For any $\bar{\omega} \in \bar{\Omega}$, we write $\bar{\omega} = (\omega, \omega^B)$, where $\omega \in \Omega$ and $\omega^B \in \Omega^B$. Also, we denote by $\bar{\mathbb{E}}$ the expectation taken under $(\bar{\Omega}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$.

For any $\varepsilon > 0$, introduce the process $Q_\varepsilon^{t,x,q}(\cdot)$ which satisfies the following dynamics

$$(2.4.9) \quad \frac{dQ_\varepsilon(s)}{Q_\varepsilon(s)} = |\theta(X^{t,x}(s))|^2 ds + \theta(X^{t,x}(s))' dW(s) + \varepsilon dB(s), \quad Q_\varepsilon^{t,x,q} = q \in (0, \infty).$$

Then under the probability space $(\bar{\Omega}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$, we have $d + 1$ risky assets, the d stocks

$X_1, \cdots, X_d$ and Q_ε. Define

$$\bar{s} := \left[\begin{array}{ccc|c} s_{11} & \cdots & s_{1d} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ s_{d1} & \cdots & s_{dd} & 0 \\ \hline \theta_1 & \cdots & \theta_d & \varepsilon \end{array}\right], \quad \bar{b} := \left[\begin{array}{c} b_1 \\ \vdots \\ b_d \\ \hline |\theta|^2 \end{array}\right],$$

and

$$\bar{a} := \bar{s}\bar{s}' = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1d} & | \\ \vdots & \ddots & \vdots & s\theta \\ a_{d1} & \cdots & a_{dd} & | \\ \hline - & \theta's' & - & |\theta|^2 + \varepsilon^2 \end{array}\right].$$

Since we assume that the matrix s has full rank (Assumption II.1), $\bar{s}$ has full rank by definition. It follows that $\bar{a}$ is positive definite. Now we can define the corresponding market price of risk under $(\bar{\Omega}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ as $\bar{\theta} := \bar{s}^{-1}\bar{b}$, and the corresponding deflator $\bar{Z}(\cdot)$ under $(\bar{\Omega}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ as the solution of

$$(2.4.10) \qquad d\bar{Z}(s) = -\bar{Z}(s)\bar{\theta}(X^{t,x}(s))'d\bar{W}(s), \quad \bar{Z}^{t,x,z}(t) = z,$$

where $\bar{W} := (W_1, \cdots, W_d, B)$ is a $(d+1)$-dimensional Brownian motion. Observe that

$$\bar{\theta} = \left[\begin{array}{c|c} s^{-1} & O_{d\times 1} \\ \hline -\frac{1}{\varepsilon}\theta's^{-1} & \frac{1}{\varepsilon} \end{array}\right]\left[\begin{array}{c} b \\ |\theta|^2 \end{array}\right] = \left[\begin{array}{c} \theta \\ 0 \end{array}\right].$$

This implies that (2.4.10) coincides with (2.4.1). Thus, we conclude that $\bar{Z}(\cdot) = Z(\cdot)$. Finally, let us introduce the function

$$\widetilde{w}_\varepsilon(t, x, q) := \bar{\mathbb{E}}[\bar{Z}^{t,x,1}(T)(Q_\varepsilon^{t,x,q}(T) - g(X^{t,x}(T)))^+],$$

for any $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$. By (2.4.9) and (2.4.3), we see that the processes $Q_\varepsilon(\cdot)$ and $Q(\cdot)$ have the following relation

$$(2.4.11) \qquad Q_\varepsilon^{t,x,q}(s) = Q^{t,x,q}(s)\exp\left\{-\frac{1}{2}\varepsilon^2(s-t) + \varepsilon(B(s) - B(t))\right\}, \quad s \in [t, T].$$

It then follows from (2.4.11), the fact that $\bar{Z}(\cdot) = Z(\cdot)$, and the definition of $\widetilde{w}_\varepsilon$ that

(2.4.12)
$$\widetilde{w}_\varepsilon(t, x, q) = \bar{\mathbb{E}}\left[\left(q\exp\left\{-\frac{1}{2}\varepsilon^2(T-t) + \varepsilon(B(T) - B(t))\right\} - Z^{t,x,1}(T)g(X^{t,x}(T))\right)^+\right].$$

Assumption II.2. *The functions θ_i and s_{ij} are locally Lipschitz, for all $i, j \in \{1, \cdots, d\}$.*

Lemma 2.4.2. *Under Assumption II.2, we have that $\widetilde{w}_\varepsilon \in \mathcal{C}^{1,2,2}((0, T) \times (0, \infty)^d \times (0, \infty))$ and satisfies the PDE*

(2.4.13) $\qquad \partial_t \widetilde{w}_\varepsilon + \frac{1}{2}Tr(\sigma\sigma' D_x^2\widetilde{w}_\varepsilon) + \frac{1}{2}(|\theta|^2 + \varepsilon^2)q^2 D_q^2\widetilde{w}_\varepsilon + q\,Tr(\sigma\theta D_{xq}\widetilde{w}_\varepsilon) = 0,$

$(t, x, q) \in (0, T) \times (0, \infty)^d \times (0, \infty)$, with the boundary condition

(2.4.14) $\qquad\qquad\qquad\qquad \widetilde{w}_\varepsilon(T, x, q) = (q - g(x))^+.$

Proof. Since $\bar{a}$ is positive definite and continuous, it must satisfy the following ellipticity condition: for every compact set $K \subset (0, \infty)^d$, there exists a positive constant C_K such that

(2.4.15) $$\sum_{i=1}^{d+1}\sum_{j=1}^{d+1}\bar{a}_{ij}(x)\xi_i\xi_j \geq C_K|\xi|^2,$$

for all $\xi \in \mathbb{R}^{d+1}$ and $x \in K$; see e.g. [57, Lemma 3]. Under Assumption II.2 and (2.4.15), the smoothness of $\widetilde{w}_\varepsilon$ and the PDE (2.4.13) follow immediately from [100, Theorem 4.7]. Finally, $\widetilde{w}_\varepsilon$ satisfies the boundary condition by definition. $\qquad\square$

Proposition 2.4.3. *For any $(t, x) \in [0, T] \times (0, \infty)^d$, the map $q \mapsto \widetilde{w}_\varepsilon(t, x, q)$ is strictly convex on $(0, \infty)$. More precisely, the map $q \mapsto D_q\widetilde{w}_\varepsilon(t, x, q)$ is strictly increasing on $(0, \infty)$ with*

$$\lim_{q\downarrow 0} D_q\widetilde{w}_\varepsilon(t, x, q) = 0, \quad and \quad \lim_{q\to\infty} D_q\widetilde{w}_\varepsilon(t, x, q) = 1.$$

Proof. We will first compute $D_q\widetilde{w}_\varepsilon(t, x, q)$, and then show that it is strictly increasing in q from 0 to 1. Let $L_\varepsilon(t, T) := \exp\left(-\frac{1}{2}\varepsilon^2(T - t) + \varepsilon(B(T) - B(t))\right)$ and $\widetilde{A}_a := \{\bar\omega : Z^{t,x,1}(T)g(X^{t,x}(T)) \le aL_\varepsilon(t, T)\}$ for $a \ge 0$. Fix $q > 0$. For any $\delta > 0$, define

$$E^\delta := \{\bar\omega : qL_\varepsilon(t, T) < Z^{t,x,1}(T)g(X^{t,x}(T)) \le (q + \delta)L_\varepsilon(t, T)\}.$$

By construction, $\widetilde{A}_q$ and E^δ are disjoint, and $\widetilde{A}_{q+\delta} = \widetilde{A}_q \cup E^\delta$. It follows that

$$\frac{1}{\delta}[\widetilde{w}_\varepsilon(t, x, q + \delta) - \widetilde{w}_\varepsilon(t, x, q)]$$

$$= \frac{1}{\delta}\left\{\bar{\mathbb{E}}\left[\left((q + \delta)L_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))\right)1_{\widetilde{A}_{q+\delta}}\right]\right.$$

$$\left. - \bar{\mathbb{E}}\left[\left(qL_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))\right)1_{\widetilde{A}_q}\right]\right\}$$

$$= \frac{1}{\delta}\left\{\bar{\mathbb{E}}\left[\left((q + \delta)L_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))\right)1_{\widetilde{A}_q}\right]\right.$$

$$+ \bar{\mathbb{E}}\left[\left((q + \delta)L_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))\right)1_{E^\delta}\right]$$

$$\left. - \bar{\mathbb{E}}\left[\left(qL_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))\right)1_{\widetilde{A}_q}\right]\right\}$$

$$= \bar{\mathbb{E}}[L_\varepsilon(t, T)1_{\widetilde{A}_q}] + \frac{1}{\delta}\bar{\mathbb{E}}\left[\left((q + \delta)L_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))\right)1_{E^\delta}\right].$$

By the definition of E^δ,

$$0 \le \frac{1}{\delta}\bar{\mathbb{E}}[\left((q + \delta)L_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))\right)1_{E^\delta}] \le \frac{1}{\delta}\bar{\mathbb{E}}[\delta L_\varepsilon(t, T)1_{E^\delta}]$$

$$= \bar{\mathbb{E}}[L_\varepsilon(t, T)1_{E^\delta}] \to 0, \text{ as } \delta \downarrow 0,$$

where we use the dominated convergence theorem. We therefore conclude that

$$D_q\widetilde{w}_\varepsilon(t, x, q) = \lim_{\delta\downarrow 0}\frac{1}{\delta}[\widetilde{w}_\varepsilon(t, x, q + \delta) - \widetilde{w}_\varepsilon(t, x, q)] = \bar{\mathbb{E}}[L_\varepsilon(t, T)1_{\widetilde{A}_q}].$$

Thanks to the dominated convergence theorem again, we have

$$\lim_{q\downarrow 0}\bar{\mathbb{E}}[L_\varepsilon(t, T)1_{\widetilde{A}_q}] = 0 \text{ and } \lim_{q\to\infty}\bar{\mathbb{E}}[L_\varepsilon(t, T)1_{\widetilde{A}_q}] = \bar{\mathbb{E}}[L_\varepsilon(t, T)] = 1.$$

It remains to prove that $D_q\widetilde{w}_\varepsilon(t, x, q) = \bar{\mathbb{E}}[L_\varepsilon(t, T)1_{\widetilde{A}_q}]$ is strictly increasing in q. Note that it is enough to show that the event E^δ has positive probability for all

$\delta > 0$. Under the integrability condition (2.2.2), the deflator $Z(\cdot)$ is strictly positive with probability 1; see e.g. [4, Section 6]. It follows from our assumptions on g (see (2.2.4) and the line before it) that $0 < Z^{t,x,1}(T)g(X^{t,x}(T)) < \infty$ $\mathbb{P}$-a.s. This implies

(2.4.16) $\qquad\qquad -\infty < \log Z^{t,x,1}(T)g(X^{t,x}(T)) < \infty \ \bar{\mathbb{P}}$-a.s.

Now, from (2.4.16) and the definitions E^δ and L_ε, we see that $\bar{\mathbb{P}}(E^\delta)$ equals to the probability of the event

$$\left\{ \bar{\omega} : \frac{\varepsilon}{2}(T - t) + \frac{1}{\varepsilon} \log \frac{Z^{t,x,1}(T)g(X^{t,x}(T))}{q + \delta} \le B(T) - B(t) \right.$$
$$\left. < \frac{\varepsilon}{2}(T - t) + \frac{1}{\varepsilon} \log \frac{Z^{t,x,1}(T)g(X^{t,x}(T))}{q} \right\}.$$

Thanks to Fubini's theorem, this probability is strictly positive. $\qquad\qquad\square$

We investigate the relation between $\widetilde{w}$ and $\widetilde{w}_\varepsilon$ in the following result.

Lemma 2.4.4. *The functions $\widetilde{w}$ and $\widetilde{w}_\varepsilon$ satisfy the following relations:*

(i) For any $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$,

$$\widetilde{w}(t, x, q) = \lim_{\varepsilon \downarrow 0} \widetilde{w}_\varepsilon(t, x, q).$$

(ii) For any compact subset $E \subset (0, \infty)$, $\widetilde{w}_\varepsilon$ converges to $\widetilde{w}$ uniformly on $[0, T] \times (0, \infty)^d \times E$. Moreover, for any $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$

(2.4.17) $\qquad\qquad \widetilde{w}(t, x, q) = \lim_{(\varepsilon, t', x', q') \to (0, t, x, q)} \widetilde{w}_\varepsilon(t', x', q').$

Proof. **(i)** By (2.4.11), we observe that

$$\bar{\mathbb{E}}\left[\sup_{\varepsilon\in(0,1]}Z^{t,x,1}(T)Q_\varepsilon^{t,x,q}(T)\right]=\bar{\mathbb{E}}\left[\sup_{\varepsilon\in(0,1]}q\exp\left\{-\frac{1}{2}\varepsilon^2(T-t)+\varepsilon(B(T)-B(t))\right\}\right]$$

$$\leq q\bar{\mathbb{E}}\left[\sup_{\varepsilon\in(0,1]}\exp\left\{\varepsilon(B(T)-B(t))\right\}\right]$$

$$\leq q\bar{\mathbb{E}}\left[\sup_{\varepsilon\in(0,1]}\exp\left\{\varepsilon(B(T)-B(t))\right\}1_{\{B(T)-B(t)\geq 0\}}\right]$$

$$+\,q\bar{\mathbb{E}}\left[\sup_{\varepsilon\in(0,1]}\exp\left\{\varepsilon(B(T)-B(t))\right\}1_{\{B(T)-B(t)<0\}}\right]$$

$$\leq q\bar{\mathbb{E}}\left[\exp\left\{B(T)-B(t)\right\}\right]+q=q\left(\exp\left\{\frac{1}{2}(T-t)\right\}+1\right)<\infty.$$

Then it follows from the dominated convergence theorem that

$$\lim_{\varepsilon\downarrow 0}\widetilde{w}_\varepsilon(t,x,q)$$

$$=\lim_{\varepsilon\downarrow 0}\bar{\mathbb{E}}\left[\left(q\exp\left\{-\frac{1}{2}\varepsilon^2(T-t)+\varepsilon(B(T)-B(t))\right\}-Z^{t,x,1}(T)g(X^{t,x}(T))\right)^+\right]$$

$$=\bar{\mathbb{E}}[(q-Z^{t,x,1}(T)g(X^{t,x}(T)))^+]=\mathbb{E}[(q-Z^{t,x,1}(T)g(X^{t,x}(T)))^+]=\widetilde{w}(t,x,q),$$

where the third equality holds as $Z^{t,x,1}(T)g(X^{t,x}(T))$ depends only on $w\in\Omega$.

(ii) From (2.4.7), (2.4.12), and the observation that $|(a-b)^+-(c-b)^+|\leq|a-c|$ for any $a,b,c\in\mathbb{R}$,

$$|\widetilde{w}_\varepsilon(t,x,q)-\widetilde{w}(t,x,q)|$$

$$\leq q\bar{\mathbb{E}}\left|\exp\left\{-\frac{1}{2}\varepsilon^2(T-t)+\varepsilon(B(T)-B(t))\right\}-1\right|$$

(2.4.18) $$\leq q\bar{\mathbb{E}}\left[\exp\left\{\frac{\varepsilon^2}{2}(T-t)+\varepsilon|B(T)-B(t)|\right\}-1\right]$$

$$=q\left[\left(1+\Phi(\varepsilon\sqrt{T-t})-\Phi(-\varepsilon\sqrt{T-t})\right)e^{\varepsilon^2(T-t)}-1\right]$$

$$\leq q\left[\left(1+\Phi(\varepsilon\sqrt{T})-\Phi(-\varepsilon\sqrt{T})\right)e^{\varepsilon^2 T}-1\right],$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Note that the second line of (2.4.18) follows from the inequality $|e^v-1|\leq e^{|v|}-1$

for $v \in \mathbb{R}$; this inequality holds because if $v < 0$, $|e^v - 1| = 1 - e^v = (e^{-v} - 1)e^v \leq e^{-v} - 1 = e^{|v|} - 1$ and if $v \geq 0$, $|e^v - 1| = e^v - 1 = e^{|v|} - 1$. We then conclude from (2.4.18) that $\widetilde{w}_\varepsilon$ converges to $\widetilde{w}$ uniformly on $[0, T] \times (0, \infty)^d \times E$, for any compact subset E of $(0, \infty)$. Now, by Lemma 2.4.2 $\widetilde{w}_\varepsilon$ is continuous on $(0, T) \times (0, \infty)^d \times (0, \infty)$. Then as a result of uniform convergence, $\widetilde{w}$ must be continuous on the same domain. Noting that

$$|\widetilde{w}_\varepsilon(t', x', q') - \widetilde{w}(t, x, q)| \leq |\widetilde{w}_\varepsilon(t', x', q') - \widetilde{w}(t', x', q')| + |\widetilde{w}(t', x', q') - \widetilde{w}(t, x, q)|,$$

we see that (2.4.17) follows from the continuity of $\widetilde{w}$ and the uniform convergence of $\widetilde{w}_\varepsilon$ to $\widetilde{w}$ on $[0, T] \times (0, \infty)^d \times E$ for any compact subset E of $(0, \infty)$. $\qquad\square$

Thanks to the stability of viscosity solutions, we have the next result immediately.

Proposition 2.4.5. *Under Assumption II.2, $\widetilde{w}$ is a continuous viscosity solution to*

$$(2.4.19) \qquad \partial_t \widetilde{w} + \frac{1}{2} Tr(\sigma\sigma' D_x^2 \widetilde{w}) + \frac{1}{2}|\theta|^2 q^2 D_q^2 \widetilde{w} + q Tr(\sigma\theta D_{xq}\widetilde{w}) = 0,$$

for $(t, x, q) \in (0, T) \times (0, \infty)^d \times (0, \infty)$, with the boundary condition

$$(2.4.20) \qquad \widetilde{w}(T, x, q) = (q - g(x))^+.$$

Proof. By Lemmas 2.4.2 and 2.4.4 (ii), the viscosity solution property follows as a direct application of [106, Proposition 2.3]. And the boundary condition holds trivially from the definition of $\widetilde{w}$. $\qquad\square$

Now we want to relate $\widetilde{w}$ to w. Given $(t, x) \in [0, T] \times (0, \infty)^d$, recall the notation in Section 2.3: for any $a \geq 0$, $\bar{A}_a := \{\omega : Z^{t,x,1}(T)g(X^{t,x}(T)) \leq a\}$; also, $F(\cdot)$ again denotes the cumulative distribution function of $Z^{t,x,1}(T)g(X^{t,x}(T))$. We first present another representation for $\widetilde{w}$ as follows.

Lemma 2.4.6. *For any $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$, we have*

$$\max_{a \geq 0} \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}] = \widetilde{w}(t, x, q).$$

Proof. Let us first take $a < q$. Since $\bar{A}_a \subset \bar{A}_q$ and $q - Z^{t,x,1}(T)g(X^{t,x}(T)) \geq 0$ on $\bar{A}_q$,

$$\mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}] \leq \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}] = \widetilde{w}(t, x, q).$$

Now consider $a > q$. Set $F := \{\omega : q < Z^{t,x,1}(T)g(X^{t,x}(T)) \leq a\}$. Observing that $\bar{A}_q$ and F are disjoint, and $\bar{A}_a = \bar{A}_q \cup F$, we have

$$\mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}]$$

$$= \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}] + \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_F]$$

$$\leq \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}] = \widetilde{w}(t, x, q),$$

where the inequality is due to the fact that $q - Z^{t,x,1}(T)g(X^{t,x}(T)) < 0$ on F. $\square$

Next, we argue that w and $\widetilde{w}$ are equal.

Proposition 2.4.7. $w(t, x, q) = \widetilde{w}(t, x, q)$, *for all* $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$.

Proof. Given $p \in [0, 1]$, there exists $a \geq 0$ such that $F(a-) \leq p \leq F(a)$. We can take two nonnegative numbers λ_1 and λ_2 with $\lambda_1 + \lambda_2 = 1$ such that

$$(2.4.21) \qquad p = \lambda_1 F(a) + \lambda_2 F(a-).$$

Observe that $p - F(a-) = \lambda_1(F(a) - F(a-))$. Plugging this into the first line of (2.3.10), we get

$$(2.4.22) \qquad U(t, x, p) = U(t, x, F(a-)) + \lambda_1 a(F(a) - F(a-)).$$

Also note from (2.3.10) that

$$a(F(a) - F(a-)) = U(t, x, F(a)) - U(t, x, F(a-)).$$

Plugging this back into (2.4.22), we obtain

$$(2.4.23) \qquad U(t, x, p) = \lambda_1 U(t, x, F(a)) + \lambda_2 U(t, x, F(a-)).$$

It then follows from (2.4.21) and (2.4.23) that

$$
\begin{aligned}
pq - U(t, x, p) &= \lambda_1[F(a)q - U(t, x, F(a))] + \lambda_2[F(a-)q - U(t, x, F(a-))] \\
(2.4.24) \qquad &\leq \max\{F(a)q - U(t, x, F(a)), F(a-)q - U(t, x, F(a-))\}.
\end{aligned}
$$

Choose a sequence $a_n \in [a/2, a)$ such that $a_n \to a$ from the left as $n \to \infty$. Thanks to Proposition 2.3.5, $p \mapsto U(t, x, p)$ is continuous on $[0, 1]$. We can therefore select a subsequence of a_n (without relabelling) such that for any $n \in \mathbb{N}$,

$$F(a-) - F(a_n) < \frac{1}{n} \quad \text{and} \quad U(t, x, F(a_n)) - U(t, x, F(a-)) < \frac{1}{n}.$$

It follows that for any $n \in \mathbb{N}$

$$F(a-)q - U(t, x, F(a-)) < F(a_n)q - U(t, x, F(a_n)) + \frac{1+q}{n},$$

which yields

$$
\begin{aligned}
F(a-)q - U(t, x, F(a-)) &\leq \limsup_{n \to \infty} \left\{ F(a_n)q - U(t, x, F(a_n)) + \frac{1+q}{n} \right\} \\
(2.4.25) \qquad &\leq \sup_{n \in \mathbb{N}} F(a_n)q - U(t, x, F(a_n)).
\end{aligned}
$$

Combining (2.4.24) and (2.4.25), we obtain

$$pq - U(t, x, p) \leq \sup_{\delta \in [a/2, a]} F(\delta)q - U(t, x, F(\delta)) \leq \sup_{\delta \geq 0} F(\delta)q - U(t, x, F(\delta)).$$

This implies

$$w(t, x, q) = \sup_{p \in [0,1]} \{pq - U(t, x, p)\} \leq \sup_{a \geq 0}\{F(a)q - U(t, x, F(a))\}.$$

Since $F(a) \in [0, 1]$ for all $a \geq 0$, the opposite inequality is trivial. As a result,

$$(2.4.26) \qquad w(t, x, q) = \sup_{p \in [0,1]} \{pq - U(t, x, p)\} = \sup_{a \geq 0}\{F(a)q - U(t, x, F(a))\}.$$

Now, thanks to (2.3.7), we have

$$
\begin{aligned}
F(a)q - U(t,x,F(a)) &= F(a)q - \mathbb{E}[Z^{t,x,1}(T)g(X^{t,x}(T))1_{\bar{A}_a}] \\
&= \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}].
\end{aligned}
$$

(2.4.27)

It follows from (2.4.26), (2.4.27) and Lemma 2.4.6 that

$$
w(t,x,q) = \max_{a \geq 0} \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}] = \widetilde{w}(t,x,q).
$$

$\square$

Remark 2.4.8. Since $w = \widetilde{w}$, we immediately have the following result from Proposition 2.4.5: w is a continuous viscosity solution to (2.4.19) on $(0,T) \times (0,\infty)^d \times (0,\infty)$ with the boundary condition (2.4.20).

2.4.3 Viscosity Supersolution Property of U

Let us extend the domain of the map $q \mapsto \widetilde{w}_\varepsilon(t,x,q)$ from $(0,\infty)$ to the entire real line $\mathbb{R}$ by setting $\widetilde{w}_\varepsilon(t,x,0) = 0$ and $\widetilde{w}_\varepsilon(t,x,q) = \infty$ for $q < 0$. In this subsection, we consider the Legendre transform of $\widetilde{w}_\varepsilon$ with respect to the q variable

$$
U_\varepsilon(t,x,p) := \sup_{q \in \mathbb{R}}\{pq - \widetilde{w}_\varepsilon(t,x,q)\} = \sup_{q \geq 0}\{pq - \widetilde{w}_\varepsilon(t,x,q)\}.
$$

We will first show that U_ε is a classical solution to a nonlinear PDE. Then we will relate U_ε to U and derive the viscosity supersolution property of U.

Proposition 2.4.9. *Under Assumption II.2, we have that* $U_\varepsilon \in \mathcal{C}^{1,2,2}((0,T) \times (0,\infty)^d \times (0,1))$ *and satisfies the equation*

$$
\begin{aligned}
0 = \partial_t U_\varepsilon &+ \frac{1}{2}Tr[\sigma\sigma' D_{xx}U_\varepsilon] + \inf_{a \in \mathbb{R}^d}\left((D_{xp}U_\varepsilon)'\sigma a + \frac{1}{2}|a|^2 D_{pp}U_\varepsilon - \theta' a D_p U_\varepsilon\right) \\
&+ \inf_{b \in \mathbb{R}^d}\left(\frac{1}{2}|b|^2 D_{pp}U_\varepsilon - \varepsilon D_p U_\varepsilon 1'b\right),
\end{aligned}
$$

(2.4.28)

where $\mathbf{1} := (1, \cdots, 1)' \in \mathbb{R}^d$, *with the boundary condition*

$$(2.4.29) \qquad\qquad U_\varepsilon(T, x, p) = pg(x).$$

Moreover, $U_\varepsilon(t, x, p)$ *is strictly convex in the* p *variable for* $p \in (0, 1)$, *with*

$$(2.4.30) \qquad \lim_{p \downarrow 0} D_p U_\varepsilon(t, x, p) = 0, \;\; and \;\; \lim_{p \uparrow 1} D_p U_\varepsilon(t, x, p) = \infty.$$

Proof. Since from Proposition 2.4.3 the function $q \mapsto D_q \widetilde{w}_\varepsilon(t, x, q)$ is strictly increasing on $(0, \infty)$ with

$$\lim_{q \downarrow 0} D_q \widetilde{w}_\varepsilon(t, x, q) = 0 \text{ and } \lim_{q \to \infty} D_q \widetilde{w}_\varepsilon(t, x, q) = 1,$$

its inverse function $p \mapsto H(t, x, p)$ is well-defined on $(0, 1)$. Moreover, considering that $\widetilde{w}_\varepsilon(t, x, q)$ is smooth on $(0, T) \times (0, \infty)^d \times (0, \infty)$, $U_\varepsilon(t, x, p)$ is smooth on $(0, T) \times (0, \infty)^d \times (0, 1)$ and can be expressed as

$$(2.4.31) \qquad U_\varepsilon(t, x, p) = \sup_{q \geq 0} \{pq - \widetilde{w}_\varepsilon(t, x, q)\} = pH(t, x, p) - \widetilde{w}_\varepsilon(t, x, H(t, x, p));$$

see e.g. [98]. By direct calculations, we have

$$D_p U_\varepsilon(t, x, p) = H(t, x, p),$$

$$D_{pp} U_\varepsilon(t, x, p) = D_p H(t, x, p) = \frac{1}{D_{qq} \widetilde{w}_\varepsilon(t, x, H(t, x, p))},$$

$$(2.4.32) \quad D_x U_\varepsilon(t, x, p) = -D_x \widetilde{w}_\varepsilon(t, x, H(t, x, p)),$$

$$D_{xx} U_\varepsilon(t, x, p) = -D_{xx} \widetilde{w}_\varepsilon(t, x, H(t, x, p)) + \frac{1}{D_{pp} U_\varepsilon(t, x, p)} (D_{px} U_\varepsilon)(D_{px} U_\varepsilon)',$$

$$D_{px} U_\varepsilon(t, x, p) = -D_{qx} \widetilde{w}_\varepsilon(t, x, H(t, x, p)) D_{pp} U_\varepsilon(t, x, p),$$

$$\partial_t U_\varepsilon(t, x, p) = -\partial_t \widetilde{w}_\varepsilon(t, x, H(t, x, p)).$$

In particular, we see that $U_\varepsilon(t, x, p)$ is strictly convex in p for $p \in (0, 1)$ and satisfies

(2.4.30). Now by setting $q := H(t, x, p)$, we deduce from (2.4.13) that

$$
\begin{aligned}
0 &= -\partial_t \widetilde{w}_\varepsilon - \frac{1}{2} Tr[\sigma\sigma' D_{xx}\widetilde{w}_\varepsilon] - \frac{1}{2}(|\theta|^2 + \varepsilon^2)q^2 D_{qq}\widetilde{w}_\varepsilon - qTr[\sigma\theta D_{xq}\widetilde{w}_\varepsilon] \\
&= \partial_t U_\varepsilon + \frac{1}{2} Tr[\sigma\sigma' D_{xx}U_\varepsilon] - \frac{1}{2D_{pp}U_\varepsilon} Tr[\sigma\sigma'(D_{px}U_\varepsilon)(D_{px}U_\varepsilon)'] \\
&\quad - \frac{1}{2}(|\theta|^2 + \varepsilon^2)\frac{(D_p U_\varepsilon)^2}{D_{pp}U_\varepsilon} + \frac{D_p U_\varepsilon}{D_{pp}U_\varepsilon} Tr[\sigma\theta D_{px}U_\varepsilon] \\
(2.4.33) \quad &= \partial_t U_\varepsilon + \frac{1}{2} Tr[\sigma\sigma' D_{xx}U_\varepsilon] + \left((D_{xp}U_\varepsilon)'\sigma a^* + \frac{1}{2}|a^*|^2 D_{pp}U_\varepsilon - \theta' a^* D_p U_\varepsilon \right) \\
&\quad + \left(\frac{1}{2}|b^*|^2 D_{pp}U_\varepsilon - \varepsilon D_p U_\varepsilon \mathbf{1}'b^* \right) \\
&= \partial_t U_\varepsilon + \frac{1}{2} Tr[\sigma\sigma' D_{xx}U_\varepsilon] + \inf_{a\in\mathbb{R}^d}\left((D_{xp}U_\varepsilon)'\sigma a + \frac{1}{2}|a|^2 D_{pp}U_\varepsilon - \theta' a D_p U_\varepsilon \right) \\
&\quad + \inf_{b\in\mathbb{R}^d}\left(\frac{1}{2}|b|^2 D_{pp}U_\varepsilon - \varepsilon D_p U_\varepsilon \mathbf{1}'b \right),
\end{aligned}
$$

where the minimizers a^* and b^* are defined by

$$
\begin{aligned}
a^*(t, x, p) &:= \frac{D_p U_\varepsilon(t, x, p)}{D_{pp}U_\varepsilon(t, x, p)}\theta(x) - \frac{1}{D_{pp}U(t, x, p)}\sigma'(x)D_{px}U_\varepsilon(t, x, p), \\
b^*(t, x, p) &:= \varepsilon\frac{D_p U_\varepsilon(t, x, p)}{D_{pp}U_\varepsilon(t, x, p)}\mathbf{1}.
\end{aligned}
$$

Finally, observe that for any $p \in (0, 1)$, the maximum of $pq - (q - g(x))^+$ is attained at $q = g(x)$. Therefore, by (2.4.14)

$$
U_\varepsilon(T, x, p) = \sup_{q\geq 0}\{pq - \widetilde{w}_\varepsilon(T, x, p)\} = \sup_{q\geq 0}\{pq - (q - g(x))^+\} = pg(x).
$$

$\square$

Now we intend to use the stability of viscosity solutions to derive the supersolution property of U. We first have the following observation.

Lemma 2.4.10. *For any $(t, x, p) \in [0, T] \times (0, \infty)^d \times \mathbb{R}$, we have*

$$
\liminf_{(\varepsilon, \tilde{t}, \tilde{x}, \tilde{p})\to(0, t, x, p)} U_\varepsilon(\tilde{t}, \tilde{x}, \tilde{p}) = U(t, x, p).
$$

Proof. As a consequence of Lemma 2.4.4 (ii), $\widetilde{w}_\varepsilon(t,x,q)$ is continuous at $(\varepsilon,t,x,q) \in [0,\infty) \times [0,T] \times (0,\infty)^d \times (0,\infty)$. This implies that $U_\varepsilon(t,x,p) = \sup_{q\geq 0}\{pq - \widetilde{w}_\varepsilon(t,x,q)\}$ is lower semicontinuous at $(\varepsilon,t,x,p) \in [0,\infty) \times [0,T] \times (0,\infty)^d \times \mathbb{R}$. It follows that

$$\liminf_{(\varepsilon,\tilde{t},\tilde{x},\tilde{p})\to(0,t,x,p)} U_\varepsilon(\tilde{t},\tilde{x},\tilde{p}) = \sup_{q\geq 0}\{pq - \widetilde{w}(t,x,q)\} = \sup_{q\geq 0}\{pq - w(t,x,q)\} = U(t,x,p),$$

where the second equality follows from Proposition 2.4.7. $\qquad\square$

Before we state the supersolution property for U, let us first introduce some notation. For any $(x,\beta,\gamma,\lambda) \in (0,\infty)^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, define

$$G(x,\beta,\gamma,\lambda) := \inf_{a\in\mathbb{R}^d}\left(\lambda'\sigma(x)a + \frac{1}{2}|a|^2\gamma - \beta\theta(x)'a\right).$$

We also consider the lower semicontinuous envelope of G

$$G_*(x,\beta,\gamma,\lambda) := \liminf_{(\tilde{x},\tilde{\beta},\tilde{\gamma},\tilde{\lambda})\to(x,\beta,\gamma,\lambda)} G(\tilde{x},\tilde{\beta},\tilde{\gamma},\tilde{\lambda}).$$

Observe that by definition,

$$(2.4.34) \qquad G_*(x,\beta,\gamma,\lambda) = \begin{cases} G(x,\beta,\gamma,\lambda), & \text{if } \gamma > 0; \\ -\infty, & \text{if } \gamma \leq 0. \end{cases}$$

Proposition 2.4.11. *Under Assumption II.2, U is a lower semicontinuous viscosity supersolution to the equation*

$$(2.4.35) \qquad 0 \geq \partial_t U + \frac{1}{2}Tr[\sigma\sigma' D_{xx}U] + G_*(x, D_pU, D_{pp}U, D_{xp}U),$$

for $(t,x,p) \in (0,T) \times (0,\infty)^d \times (0,1)$, with the boundary condition

$$(2.4.36) \qquad U(T,x,p) = pg(x),$$

Proof. Note that the lower semicontinuity of U is a consequence of Lemma 2.4.10, and the boundary condition (2.4.36) comes from the fact that $w = \widetilde{w}$ and the definition of $\widetilde{w}$, as the following calculation demonstrates:

$$U(T, x, p) = \sup_{q \geq 0}\{pq - w(T, x, p)\} = \sup_{q \geq 0}\{pq - \widetilde{w}(T, x, p)\}$$

$$= \sup_{q \geq 0}\{pq - (q - g(x))^+\} = pg(x).$$

Let us now turn to the PDE characterization inside the domain of U. Set $\bar{x} := (t, x, p)$. Let φ be a smooth function such that $U - \varphi$ attains a local minimum at $\bar{x}_0 = (t_0, x_0, p_0) \in (0, T) \times (0, \infty)^d \times (0, 1)$ and $U(\bar{x}_0) = \varphi(\bar{x}_0)$. Note from (2.4.34) that as $D_{pp}\varphi(\bar{x}_0) \leq 0$, we must have $G_*(x_0, D_p\varphi, D_{pp}\varphi, D_{xp}\varphi) = -\infty$. Thus, the viscosity supersolution property (2.4.35) is trivially satisfied. We therefore assume in the following that $D_{pp}\varphi(\bar{x}_0) > 0$.

Let $F_\varepsilon(\bar{x}, \partial_t U_\varepsilon(\bar{x}), D_p U_\varepsilon(\bar{x}), D_{pp}U_\varepsilon(\bar{x}), D_{xp}U_\varepsilon(\bar{x}), D_{xx}U_\varepsilon(\bar{x}))$ denote the right hand side of (2.4.28). Observe from the calculation in (2.4.33) that as $\gamma > 0$,

$$F_\varepsilon(\bar{x}, \alpha, \beta, \gamma, \lambda, A) = \alpha + \frac{1}{2}Tr[\sigma(x)\sigma(x)'A]$$

$$- \frac{1}{2\gamma}Tr[\sigma(x)\sigma(x)'\lambda\lambda'] - \frac{\beta^2}{2\gamma}(|\theta(x)|^2 + \varepsilon^2) + \frac{\beta}{\gamma}Tr[\sigma(x)\theta(x)\lambda].$$

This shows that F_ε is continuous at every $(\varepsilon, \bar{x}, \alpha, \beta, \gamma, \lambda, A)$ as long as $\gamma > 0$. It follows that for any $z = (\bar{x}, \alpha, \beta, \gamma, \lambda, A)$ with $\gamma > 0$, we have

(2.4.37)

$$F_*(z) := \liminf_{(\varepsilon, z') \to (0, z)} F_\varepsilon(z') = F_0(z)$$

$$= \alpha + \frac{1}{2}Tr[\sigma(x)\sigma(x)'A] + \inf_{a \in \mathbb{R}^d}\left(\lambda'\sigma(x)a + \frac{1}{2}|a|^2\gamma - \theta(x)'a\beta\right).$$

Since we have $U(\bar{x}) = \liminf_{(\varepsilon, \bar{x}') \to (0, \bar{x})} U_\varepsilon(\bar{x}')$ from Lemma 2.4.10, we may use the same argument in [106, Proposition 2.3] and obtain that

$$F_*(\bar{x}_0, \partial_t\varphi(\bar{x}_0), D_p\varphi(\bar{x}_0), D_{pp}\varphi(\bar{x}_0), D_{xp}\varphi(\bar{x}_0), D_{xx}\varphi(\bar{x}_0)) \leq 0.$$

Considering that $D_{pp}\varphi(\bar{x}_0) > 0$, we see from (2.4.37) and (2.4.34) that this is the desired supersolution property. □

A few remarks are in order:

Remark 2.4.12. Results similar to Proposition 2.4.11 were proved by [22], with stronger assumptions (such as the existence of an equivalent martingale measure and the existence of a unique strong solution to (2.2.1)), using the stochastic target formulation. Here, we first observe that the Legendre transform of U is equal to $\widetilde{w}$ and that $\widetilde{w}$ can be approximated by $\widetilde{w}_\varepsilon$, which is a classical solution to a linear PDE and is strictly convex in q; then, we apply the Legendre duality argument, as carried out in [65], to show that U_ε, the Legendre transform of $\widetilde{w}_\varepsilon$, is a classical solution to a nonlinear PDE. Finally, the stability of viscosity solutions leads to the viscosity supersolution property of U.

Remark 2.4.13. Instead of relying on the Legendre duality we could directly apply the dynamic programming principle of [56] for weak solutions to the formulation in Section 2.3.1. The problem with this approach is that it requires some growth conditions on the coefficients of (2.2.1), which would rule out the possibility of arbitrage, the thing we are interested in and want to keep in the scope of our discussion.

Remark 2.4.14. Under our assumptions, the solution of (2.4.35) may not be unique as pointed out below.

(i) Let us consider the PDE satisfied by the superhedging price $U(t, x, 1)$:

$$(2.4.38) \qquad 0 = v_t + \frac{1}{2}\mathrm{Tr}(\sigma\sigma'D_x^2 v), \quad \text{on } (0, T) \times (0, \infty)^d,$$

$$(2.4.39) \qquad v(T-, x) = g(x), \quad \text{on } (0, \infty)^d.$$

Unless additional boundary conditions are specified, this PDE may have multiple solutions. The role of additional boundary conditions in identifying $(t, x) \rightarrow$

$U(t, x, 1)$ as the unique solution of the above Cauchy problem is discussed in Section 4 of [12]. Also see [94] for a similar discussion on boundary conditions for degenerate parabolic problems on bounded domains.

Even when additional boundary conditions are specified, the growth of σ might lead to the loss of uniqueness; see for example [13] and Theorem 4.8 of [12] which give necessary and sufficient conditions on the uniqueness of Cauchy problems in one and two dimensional setting in terms of the growth rate of its coefficients. We also note that [41] develops necessary and sufficient conditions for uniqueness, in terms of the attainability of the boundary of the positive orthant by an auxiliary diffusion (or, more generally, an auxiliary Itô) process.

(ii) Let $\Delta U(t, x, 1)$ be the difference of two solutions of (2.4.38)-(2.4.39). Then both $U(t, x, p)$ and $U(t, x, p) + \Delta U(t, x, 1)$ are solutions of (2.4.35) (along with its boundary conditions). As a result, whenever (2.4.38) and (2.4.39) has multiple solutions, so does the PDE (2.4.35) for the value function U.

2.4.4 Characterizing the value function U

We intend to characterize U_ε as the smallest solution among a particular class of functions, as specified below in Proposition 2.4.15. Then, considering that

$$\liminf_{(\varepsilon, \tilde{t}, \tilde{x}, \tilde{p}) \to (0, t, x, p)} U_\varepsilon(\tilde{t}, \tilde{x}, \tilde{p}) = U(t, x, p)$$

from Lemma 2.4.10, this gives a characterization for U. In determining U numerically, one could use U_ε as a proxy for U for small enough ε.

Additionally, we will also characterize U as the smallest nonnegative supersolution of (2.4.35) in Proposition 2.4.16.

Proposition 2.4.15. *Suppose that Assumption II.2 holds. Let* $u : [0, T] \times (0, \infty)^d \times [0, 1] \mapsto [0, \infty)$ *be of class* $\mathcal{C}^{1,2,2}((0, T) \times (0, \infty)^d \times (0, 1))$ *such that* $u(t, x, 0) = 0$ *and*

$u(t, x, p)$ is strictly convex in p for $p \in (0, 1)$ with

(2.4.40)
$$\lim_{p\downarrow 0} D_p u(t, x, p) = 0 \ \text{ and } \ \lim_{p\uparrow 1} D_p u(t, x, p) = \infty.$$

If u satisfies the following partial differential inequality

(2.4.41)
$$0 \geq \partial_t u + \frac{1}{2}Tr[\sigma\sigma' D_{xx}u] + \inf_{a\in\mathbb{R}^d}\left((D_{xp}u)'\sigma a + \frac{1}{2}|a|^2 D_{pp}u - \theta' a D_p u\right)$$
$$+ \inf_{b\in\mathbb{R}^d}\left(\frac{1}{2}|b|^2 D_{pp}u - \varepsilon D_p u \mathbf{1}'b\right),$$

where $\mathbf{1} := (1, \cdots, 1)' \in \mathbb{R}^d$, with the boundary condition

(2.4.42)
$$u(T, x, p) = pg(x),$$

then $u \geq U_\varepsilon$.

Proof. Let us extend the domain of the map $p \mapsto u(t, x, p)$ from $[0, 1]$ to the entire real line $\mathbb{R}$ by setting $u(t, x, p) = 0$ for $p < 0$ and $u(t, x, p) = \infty$ for $p > 1$. Then, we can define the Legendre transform of u with respect to the p variable

(2.4.43) $\quad w^u(t, x, q) := \sup_{p\in\mathbb{R}}\{pq - u(t, x, p)\} = \sup_{p\in[0,1]}\{pq - u(t, x, p)\} \geq 0, \ \text{for } q \geq 0,$

where the positivity comes from the condition $u(t, x, 0) = 0$. First, observe that since u is nonnegative, we must have

(2.4.44)
$$w^u(t, x, q) \leq \sup_{p\in[0,1]} pq = q, \ \text{for any } q \geq 0.$$

Next, we derive the boundary condition of w^u from (2.4.42) as

(2.4.45) $\quad w^u(T, x, q) = \sup_{p\in[0,1]}\{pq - u(T, x, p)\} = \sup_{p\in[0,1]}\{pq - pg(x)\} = (q - g(x))^+.$

Now, since $u(t, x, p)$ is strictly convex in p for $p \in (0, 1)$ and satisfies (2.4.40), we can express w^u as

$$w^u(t, x, q) = J(t, x, q)q - u(t, x, J(t, x, q)), \text{for } q \in (0, \infty),$$

where $q \mapsto J(\cdot, q)$ is the inverse function of $p \mapsto D_p u(\cdot, p)$. We can therefore compute the derivatives of $w^u(t, x, q)$ in terms of those of $u(t, x, J(t, x, q))$, as carried out in (2.4.32). We can then perform the same calculation in (2.4.33) (but going backward), and deduce from (2.4.41) that for any $(t, x, q) \in (0, T) \times (0, \infty)^d \times (0, \infty)$,

$$(2.4.46) \qquad 0 \leq \partial_t w^u + \frac{1}{2} Tr[\sigma\sigma' D_{xx} w^u] + \frac{1}{2}(|\theta|^2 + \varepsilon^2)q^2 D_{qq} w^u + q Tr[\sigma\theta D_{xq} w^u].$$

Define the process $Y(s) := Z^{t,x,1}(s) Q_\varepsilon^{t,x,q}(s)$ for $s \in [t, T]$. Observing that

$$Y(s) = q \exp\{-\frac{1}{2}\varepsilon^2(s - t) + \varepsilon(B(s) - B(t))\},$$

we conclude $Y(\cdot)$ is a martingale with $\bar{\mathbb{E}}[Y(s)] = q$ and $\mathrm{Var}(Y(s)) = q^2(e^{\varepsilon^2(s-t)} - 1)$ for all $s \in [t, T]$, and satisfies the following SDE

$$dY(s) = \varepsilon Y(s) dB(s) \text{ for } s \in [t, T], \text{ and } Y(t) = q.$$

Thanks to the Burkholder-Davis-Gundy inequality, there is some $C > 0$ such that

$$(2.4.47) \qquad \begin{aligned} \bar{\mathbb{E}}\left[\max_{t \leq s \leq T} |Y(s)|^2\right] &\leq C\bar{\mathbb{E}}\left[\int_t^T \varepsilon^2 Y^2(s) ds\right] \\ &= C\varepsilon^2 \int_t^T q^2(e^{\varepsilon^2(s-t)} - 1) + q^2 \ ds < \infty. \end{aligned}$$

For all $n \in \mathbb{N}$, define the stopping time $\tau_n := \inf\{s \geq t : |X^{t,x}(s)| > n \text{ or } |Q_\varepsilon^{t,x,q}(s)| > n\}$. By applying the product rule to the process $Z^{t,x,1}(\cdot)w^u(\cdot, X^{t,x}(\cdot), Q_\varepsilon^{t,x,q}(\cdot))$ and using (2.4.46), we obtain that for all $n \in \mathbb{N}$,

$$(2.4.48) \qquad w^u(t, x, q) \leq \bar{\mathbb{E}}[Z^{t,x,1}(T \wedge \tau_n)w^u(T \wedge \tau_n, X^{t,x}(T \wedge \tau_n), Q_\varepsilon^{t,x,q}(T \wedge \tau_n))].$$

Now, observe from (2.4.44) that $Z^{t,x,1}(s)w^u(s, X^{t,x}(s), Q_\varepsilon^{t,x,q}(s)) \leq Y(s)$ for any $s \in [t, T]$. Then from (2.4.47), we may apply the dominated convergence theorem to (2.4.48) and obtain

$$w^u(t, x, q) \leq \bar{\mathbb{E}}[Z^{t,x,1}(T)w^u(T, X^{t,x}(T), Q_\varepsilon^{t,x,q}(T))]$$

$$= \bar{\mathbb{E}}[Z^{t,x,1}(T)(Q_\varepsilon^{t,x,q}(T) - g(X^{t,x}(T)))^+] = \widetilde{w}_\varepsilon(t, x, q),$$

where the first equality is due to (2.4.45). It follows that

$$u(t, x, p) = \sup_{q \geq 0}\{pq - w^u(t, x, q)\} \geq \sup_{q \geq 0}\{pq - \widetilde{w}_\varepsilon(t, x, q)\} = U_\varepsilon(t, x, p).$$

$\square$

Proposition 2.4.16. *Suppose Assumption II.2 holds. Let $u : [0, T] \times (0, \infty)^d \times [0, 1] \mapsto [0, \infty)$ be such that $u(t, x, 0) = 0$, $u(t, x, p)$ is convex in p, and the Legendre transform of u with respect to the p variable, as defined in the proof of Proposition 2.4.15, is continuous on $[0, T] \times (0, \infty)^d \times (0, \infty)$. If u is a lower semicontinuous viscosity supersolution to (2.4.35) on $(0, T) \times (0, \infty)^d \times (0, 1)$ with the boundary condition (2.4.36), then $u \geq U$.*

Proof. Let us denote by w^u the Legendre transform of u with respect to p. By the same argument in the proof of Proposition 2.4.15, we can show that (2.4.43), (2.4.44) and (2.4.45) are true. Moreover, as demonstrated in [22, Section 4], by using the supersolution property of u we may show that w^u is an upper semicontinuous viscosity subsolution on $(0, T) \times (0, \infty)^d \times (0, \infty)$ to the equation

$$(2.4.49) \qquad \partial_t w^u + \frac{1}{2}Tr(\sigma\sigma' D_x^2 w^u) + \frac{1}{2}|\theta|^2 q^2 D_q^2 w^u + qTr(\sigma\theta D_{xq}w^u) = 0.$$

Let $\rho(t, x, q)$ be a nonnegative $\mathcal{C}^\infty$ function supported in $\{(t, x, q) : t \in [0, 1], |(x, q)| \leq 1\}$ with unit mass. Without loss of generality, set $w^u(t, x, q) = 0$ for $(t, x, q) \in \mathbb{R}^{d+2} \cap \left([0, T] \times (0, \infty)^d \times (0, \infty)\right)^c$. Then for any $(t, x, q) \in \mathbb{R}^{d+2}$, define

$$w^u_\delta(t, x, q) := \rho^\delta * w^u \text{ where } \rho^\delta(t, x, q) := \frac{1}{\delta^{d+2}}\rho\left(\frac{t}{\delta^2}, \frac{x}{\delta}, \frac{q}{\delta}\right).$$

By definition, w^u_δ is $\mathcal{C}^\infty$. Moreover, it can be shown that w^u_δ is a subsolution to (2.4.49) on $(0, T) \times (0, \infty)^d \times (0, \infty)$; see e.g. (3.23)-(3.24) in [39, Section 3.3.2] and [5, Lemma 2.7]. Set $\bar{x} = (t, x, q)$. By (2.4.44), we see from the definition of w^u_δ that

$$(2.4.50) \qquad w^u_\delta(\bar{x}) = \int_{\mathbb{R}^{d+2}} \rho^\delta(y)w^u(\bar{x} - y)dy \leq (q + \delta) \int_{\mathbb{R}^{d+2}} \rho^\delta(y)dy = q + \delta.$$

Also, the continuity of w^u implies that $w^u_\delta \to w^u$ for every $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$. Since w^u_δ is a classical subsolution to (2.4.49), we have for all $n \in \mathbb{N}0$ that

$$(2.4.51) \qquad w^u_\delta(t, x, q) \leq \mathbb{E}[Z^{t,x,1}(T \wedge \tau_n) w^u_\delta(T \wedge \tau_n, X^{t,x}(T \wedge \tau_n), Q^{t,x,q}(T \wedge \tau_n))],$$

where $\tau_n := \inf\{s \geq t : |X^{t,x}(s)| > n \text{ or } |Q^{t,x,q}(s)| > n\}$. For each fixed $n \in \mathbb{N}$, thanks to (2.4.50) we may apply the dominated convergence theorem as we take the limit $\delta \to 0$ in (2.4.51). We thus get

$$(2.4.52) \qquad w^u(t, x, q) \leq \mathbb{E}[Z^{t,x,1}(T \wedge \tau_n) w^u(T \wedge \tau_n, X^{t,x}(T \wedge \tau_n), Q^{t,x,q}(T \wedge \tau_n))].$$

Now by applying the Reverse Fatou's Lemma (see e.g. [110, p.53]) to (2.4.52),

$$
\begin{aligned}
w^u(t, x, q) \;&\leq\; \mathbb{E}[Z^{t,x,1}(T) \limsup_{n \to \infty} w^u(T \wedge \tau_n, X^{t,x}(T \wedge \tau_n), Q^{t,x,q}(T \wedge \tau_n))] \\
&\leq\; \mathbb{E}[Z^{t,x,1}(T) w^u(T, X^{t,x}(T), Q^{t,x,q}(T))] \\
&\leq\; \mathbb{E}[Z^{t,x,1}(T)(Q^{t,x,q}(T) - g(X^{t,x}(T)))^+] = w(t, x, q),
\end{aligned}
$$

where the second inequality follows from the upper semicontinuity of w^u and the third inequality is due to (2.4.45). Finally, we conclude that

$$u(t, x, p) = \sup_{q \geq 0}\{pq - w^u(t, x, q)\} \geq \sup_{q \geq 0}\{pq - w(t, x, q)\} = U(t, x, p),$$

where the first equality is due to the convexity and the lower semicontinuity of u. $\quad\square$

One should note that U_ε and U satisfy the assumptions stated in Propositions 2.4.15 and 2.4.16, respectively. Therefore, one can indeed see these results as PDE characterizations of the functions U_ε and U.

In this chapter, under the context where equivalent martingale measures need not exist, we discuss the quantile hedging problem and focus on the PDE characterization for the minimum amount of initial capital required for quantile hedging. An interesting problem following this is the construction of the corresponding quantile hedging portfolio. We leave this problem open for future research.

CHAPTER III

Robust Maximization of Asymptotic Growth under Covariance Uncertainty

3.1 Introduction

In this chapter, we consider the problem of how to trade optimally in a market when the investing horizon is long and the dynamics of the underlying assets are uncertain. For the case where the uncertainty lies only in the instantaneous expected return of the underlying assets, this problem has been studied by Kardaras & Robertson [73]. They identify the optimal trading strategy using a generalized version of the principle eigenfunction for a linear elliptic operator which depends on the given covariance structure of the underlying assets. We intend to generalize their results to the case where *even* the covariance structure of the underlying assets is not known precisely, which is suggested in [73, Discussion]. More precisely, we would like to determine a *robust* trading strategy under which the asymptotic growth rate of one's wealth, defined below, can be maximized no matter which admissible covariance structure materializes.

Uncertainty in variance (or, equivalently, in covariance) has been drawing increasing attention. The main difficulty lies in the absence of one single dominating probability measure among Π, the collection of all probability measures induced by variance uncertainty. In their pioneering works, Avellaneda, Levy & Paras [3] and

Lyons [81] introduced the uncertain volatility model (UVM), where the volatility process is only known to lie in a fixed interval $[\underline{\sigma}, \overline{\sigma}]$. Under the Markovian framework, they obtained a duality formula for the superhedging price of (non-path-dependent) European contingent claims. Under a generalized version of the UVM, Denis & Martini [32] extended the above duality formula, by using the capacity theory, to incorporate path-dependent European contingent claims. For the capacity theory to work, they required some continuity of the random variables being hedged. Taking a different approach based on the underlying partial differential equations, Peng [89] derived results very similar to [32]. The connection between [32] and [89] was then elaborated and extended in Denis, Hu & Peng [31]. On the other hand, instead of imposing some continuity assumptions on the random variables being hedged, Soner, Touzi & Zhang [103] chose to restrict slightly the collection of non-dominated probability measures, and derived under this setting a duality formulation for the superhedging problem. With all these developments, superhedging under volatility uncertainty has then been further studied in Nutz & Soner [88] and Nutz [87], among others. Also notice that Fernholz & Karatzas [43] characterized the highest return relative to the market portfolio under covariance uncertainty. Moreover, a controller-and-stopper game with controlled drift and volatility is considered in Chapter IV, which can be viewed as an optimal stopping problem under volatility uncertainty.

While we also take covariance uncertainty into account, we focus on robust growth-optimal trading, which is different by nature from the superhedging problem. Here, an investor intends to find a trading strategy such that her wealth process can achieve maximal growth rate, in certain sense, uniformly over all possible probability measures in Π, or at least in a large enough subset Π^* of Π. Previous research on this problem can be found in [73] and the references therein. It is worth noting that this

problem falls under the umbrella of ergodic control, for which the dynamic programming heuristic cannot be directly applied; see e.g. Arapostathis, Borkar & Ghosh [2] and Borkar [19], where they consider ergodic control problems with controlled drift.

Following the framework in [73], we first observe that the associated differential operator under covariance uncertainty is a variant of Pucci's extremal operator. We define the "principal eigenvalue" for this fully nonlinear operator, denoted by λ^*, in some appropriate sense, and then investigate the connection between λ^* and the generalized principal eigenvalue in [73] where the covariance structure is a priori given. This connection is first established on smooth bounded domains, thanks to the theory of continuous selection in Michael [85] and Brown [25]. Next, observing that a Harnack inequality holds under current context, we extend the result to unbounded domains. Finally, as a consequence of this connection, we generalize [73, Theorem 2.1] to the case with covariance uncertainty: we characterize the largest possible asymptotic growth rate as λ^* (which is robust among probabilities in a large enough subset Π^* of Π) and identify a robust trading strategy in terms of λ^* and the corresponding eigenfunction; see Theorem 3.3.14.

The structure of this chapter is as follows. In Section 3.2, we introduce the framework of our study and formulate the problem of robust maximization of asymptotic growth under covariance uncertainty. In Section 3.3, we first introduce several different notions of the generalized principal eigenvalue, and then investigate the relation between them. The main technical result we obtain is Theorem 3.3.12, using which we resolve the problem of robust maximization of asymptotic growth in Theorem 3.3.14.

3.1.1 Notation

We collect some notation and definitions here for readers' convenience.

- $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$, and *Leb* denotes Lebesgue measure in

$\mathbb{R}^n$.

- $B_\delta(x)$ denotes the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $\delta > 0$.

- $\bar{D}$ denotes the closure of D and ∂D denotes the boundary of D.

- Given $x \in \mathbb{R}^n$ and $D_1, D_2 \subset \mathbb{R}^n$, $d(x, D_1) := \inf\{|x - y| \mid y \in D_1\}$ and $d(D_1, D_2) := \inf\{|x - y| \mid x \in D_1, y \in D_2\}$.

- Given $D \subset \mathbb{R}^n$, $C(D) = C^0(D)$ denotes the set of continuous functions on D. If D is open, $C^k(D)$ denotes the set of functions having derivatives of order $\leq k$ continuous in D, and $C^k(\bar{D})$ denotes the set of functions in $C^k(D)$ whose derivatives of order $\leq k$ have continuous extension on $\bar{D}$.

- Given $D \subset \mathbb{R}^n$, $C^{k,\beta}(D)$ denotes the set of functions in $C^k(D)$ whose derivatives of order $\leq k$ are Holder continuous on D with exponent $\beta \in (0, 1]$. Moreover, $C^{k,\beta}_{\text{loc}}(D)$ denotes the set of functions belonging to $C^{k,\beta}(K)$ for every compact subset K of D.

- We say $D \subset \mathbb{R}^n$ is a domain if it is an open connected set. We say D is a smooth domain if it is a domain whose boundary is of $C^{2,\beta}$ for some $\beta \in (0, 1]$.

- Given $D \subset \mathbb{R}^n$ and $u : D \mapsto \mathbb{R}$, $\underset{D}{\operatorname{osc}} := \sup\{|u(x) - u(y)| \mid x, y \in D\}$.

3.2   The Set-Up

Fix $d \in \mathbb{N}$. Consider an open connected set $E \subseteq \mathbb{R}^d$, and two functions $\theta, \Theta : E \mapsto (0, \infty)$. The following assumption will be in force throughout this chapter.

Assumption III.1.   *(i) θ and Θ are of $C^{0,\alpha}_{\text{loc}}(E)$ for some $\alpha \in (0, 1]$, and $\theta < \Theta$ in E.*

(ii) There is a sequence $\{E_n\}_{n \in \mathbb{N}}$ of bounded open convex subsets of E such that ∂E_n is of $C^{2,\alpha'}$ for some $\alpha' \in (0, 1]$, $\bar{E}_n \subset E_{n+1}$ for all $n \in \mathbb{N}$, and $E = \bigcup_{n=1}^{\infty} E_n$.

Let $\mathbb{S}^d$ denote the space of $d \times d$ symmetric matrices, equipped with the norm

$$(3.2.1) \qquad \|M\| := \max_{i=1,\cdots,d} |e_i(M)|, \ M \in \mathbb{S}^d,$$

where $e_i(M)$'s are the eigenvalues of M. In some cases, we will also consider the norm $\|M\|_{\max} := \max |m_{ij}|$, for $M = \{m_{ij}\}_{i,j} \in \mathbb{S}^d$. These two norms are equivalent with $\| \cdot \|_{\max} \leq \| \cdot \| \leq d\| \cdot \|_{\max}$.

Definition 3.2.1. Let $\mathcal{C}$ be the collection of functions $c : E \mapsto \mathbb{S}^d$ such that

(i) for any $x \in E$, $\theta(x)|\xi|^2 \leq \xi' c(x)\xi \leq \Theta(x)|\xi|^2$, $\forall \, \xi \in \mathbb{R}^d \setminus \{0\}$;

(ii) $c_{ij}(x)$ is of $C^{1,\alpha}_{\text{loc}}(E)$, $1 \leq i,j \leq d$.

Let $\widehat{E} := E \cup \triangle$ be the one-point compactification of E, where $\triangle$ is identified with ∂E if E is bounded with ∂E plus the point at infinity if E is unbounded. Following the set-up in [73, Section 1] or [92, p.40], we consider the space $C([0,\infty), \widehat{E})$ of continuous functions $\omega : [0,\infty) \mapsto \widehat{E}$, and define for each $\omega \in C([0,\infty), \widehat{E})$ the exit times

$$\zeta_n(\omega) := \inf\{t \geq 0 \mid \omega_t \notin E_n\}, \quad \zeta(\omega) := \lim_{n \to \infty} \zeta_n(\omega).$$

Then, we introduce $\Omega := \left\{\omega \in C([0,\infty), \widehat{E}) \ \middle| \ \omega_{\zeta+t} = \triangle \text{ for all } t \geq 0, \text{ if } \zeta(\omega) < \infty\right\}$. Let $X = \{X_t\}_{t \geq 0}$ be the coordinate mapping process for $\omega \in \Omega$. Set $\{\mathcal{B}_t\}_{t \geq 0}$ to be the natural filtration generated by X, and denote by $\mathcal{B}$ the smallest σ-algebra generated by $\bigcup_{t \geq 0} \mathcal{B}_t$. Similarly, set $(\mathcal{F}_t)_{t \geq 0}$ to be the right-continuous enlargement of $(\mathcal{B}_t)_{t \geq 0}$, and denote by $\mathcal{F}$ the smallest σ-algebra generated by $\bigcup_{t \geq 0} \mathcal{F}_t$.

Remark 3.2.2. For financial applications, $X = \{X_t\}_{t \geq 0}$ represents the (relative) price process of certain underlying assets, and each $c \in \mathcal{C}$ represents a possible covariance structure that might eventually materialize. In view of Definition 3.2.1 (i), the extent of the uncertainty in covariance is captured by the functions θ and

Θ: they act as the pointwise lower and upper bounds uniformly over all possible covariance structures $c \in \mathcal{C}$.

3.2.1 The generalized martingale problem

For any $M = \{m_{ij}\}_{i,j} \in \mathbb{S}^d$, define the operator L^M which acts on $f \in C^2(E)$ by

$$(L^M f)(x) := \frac{1}{2} \sum_{i,j=1}^{d} m_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{1}{2} \text{Tr}[MD^2 f(x)], \quad x \in E.$$

For each $c \in \mathcal{C}$, we define similarly the operator $L^{c(\cdot)}$ as

$$(L^{c(\cdot)} f)(x) := \frac{1}{2} \sum_{i,j=1}^{d} c_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{1}{2} \text{Tr}[c(x)D^2 f(x)], \quad x \in E.$$

Given $c \in \mathcal{C}$, a solution to the generalized martingale problem on E for the operator $L^{c(\cdot)}$ is a family of probability measures $(\mathbb{Q}_x^c)_{x \in \widehat{E}}$ on $(\Omega, \mathcal{B})$ such that $\mathbb{Q}_x^c[X_0 = x] = 1$ and

$$f(X_{s \wedge \zeta_n}) - \int_0^{s \wedge \zeta_n} (L^{c(\cdot)} f)(X_u) du$$

is a $(\Omega, (\mathcal{B}_t)_{t \geq 0}, \mathbb{Q}_x^c)$-martingale for all $n \in \mathbb{N}$ and $f \in C^2(E)$.

The following result, taken from [92, Theorem 1.13.1], states that Assumption III.1 guarantees the existence and uniqueness of the solutions to the generalized martingale problem on E for the operator $L^{c(\cdot)}$, for each fixed $c \in \mathcal{C}$.

Proposition 3.2.3. *Under Assumption III.1, for each $c \in \mathcal{C}$, there is a unique solution $(\mathbb{Q}_x^c)_{x \in \widehat{E}}$ to the generalized martingale problem on E for the operator $L^{c(\cdot)}$.*

Remark 3.2.4. For each $c \in \mathcal{C}$, as mentioned in [73, Section 1],

$$f(X_{s \wedge \zeta_n}) - \int_0^{s \wedge \zeta_n} (L^{c(\cdot)} f)(X_u) du$$

is also a $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q}_x^c)$-martingale for all $n \in \mathbb{N}$ and $f \in C^2(E)$, as f and $L^{c(\cdot)} f$ are bounded in each E_n. Now, by taking $f(x) = x^i$, $i = 1, \cdots, d$ and $f(x) = x^i x^j$ with $i, j, = 1 \cdots d$, we get $X_{t \wedge \zeta_n}$ is a $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q}_x^c)$-martingale with quadratic covariation process $\int_0^{\cdot} 1_{\{t \leq \zeta_n\}} c(X_t) dt$, for each $n \in \mathbb{N}$ and $x \in \widehat{E}$.

3.2.2 Asymptotic growth rate

For any fixed $x_0 \in E$, we will simply write $\mathbb{Q}^c = \mathbb{Q}^c_{x_0}$ for all $c \in \mathcal{C}$, when there is no confusion on the initial value x_0 of X. Let us denote by Π the collection of probability measures on $(\Omega, \mathcal{F})$ which are locally absolutely continuous with respect to $\mathbb{Q}^c$ (written $\mathbb{P} \ll_{\mathrm{loc}} \mathbb{Q}^c$) for some $c \in \mathcal{C}$, and for which the process X does not explode. That is,

$$\Pi := \{\mathbb{P} \in P(\Omega, \mathcal{F}) \mid \exists\, c \in \mathcal{C} \text{ s.t. } \mathbb{P}|_{\mathcal{F}_t} \ll \mathbb{Q}^c|_{\mathcal{F}_t} \text{ for all } t \geq 0, \text{ and } \mathbb{P}[\zeta < \infty] = 0\},$$

where $P(\Omega, \mathcal{F})$ denotes the collection of all probability measures on $(\Omega, \mathcal{F})$. As observed in [73, Section 1], for each $\mathbb{P} \in \Pi$, X is a $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$-semimartingale such that $\mathbb{P}[X \in C([0, \infty), E)] = 1$. Moreover, if we take $c \in \mathcal{C}$ such that $\mathbb{P} \ll_{\mathrm{loc}} \mathbb{Q}^c$, then X admits the representation

$$X_\cdot = x_0 + \int_0^\cdot b_t^{\mathbb{P}} dt + \int_0^\cdot \sigma(X_t) dW_t^{\mathbb{P}},$$

where $W^{\mathbb{P}}$ is a standard d-dimensional Brownian motion on $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, σ is the unique symmetric strictly positive definite square root of c, and $b^{\mathbb{P}}$ is a d-dimensional $\{F_t\}_{t \geq 0}$-progressively measurable process.

Let $(Z_t)_{t \geq 0}$ be an adapted process. For $\mathbb{P} \in \Pi$, define

$$\mathbb{P}\text{-}\liminf_{t \to \infty} Z_t := \operatorname{ess\,sup}^{\mathbb{P}} \left\{ \chi \text{ is } \mathcal{F}\text{-measurable} \,\Big|\, \lim_{t \to \infty} \mathbb{P}[Z_t \geq \chi] = 1 \right\}.$$

For any d-dimensional predictable process π which is X-integrable under $\mathbb{Q}^c$ for all $c \in \mathcal{C}$, we can define the process $V_\cdot^\pi := 1 + \int_0^\cdot \pi_t' dX_t$ under $\mathbb{Q}^c$ for all $c \in \mathcal{C}$. Let $\mathcal{V}$ denote the collection of all such processes π which in addition satisfy the following: for each $c \in \mathcal{C}$, $\mathbb{Q}^c[V_t^\pi > 0] = 1 \,\forall\, t \geq 0$. Here, $\pi \in \mathcal{V}$ represents an admissible trading strategy and V^π represents the corresponding wealth process. Now, for any $\pi \in \mathcal{V}$,

we define the asymptotic growth rate of V^π under $\mathbb{P} \in \Pi$ as

$$g(\pi; \mathbb{P}) := \sup \left\{ \gamma \in \mathbb{R} \ \middle| \ \mathbb{P}\text{-}\liminf_{t \to \infty}(t^{-1} \log V_t^\pi) \geq \gamma, \ \mathbb{P}\text{-a.s.} \right\}.$$

3.2.3   The problem

The problem we consider in this chapter is how to choose a trading strategy $\pi^* \in \mathcal{V}$ such that the wealth process V^{π^*} attains the robust maximal asymptotic growth rate under all possible probabilities in Π, or at least, in a large enough subset of Π which readily contains all "non-pathological" cases. More precisely, in Theorem 3.3.14 below, we will construct a large enough suitable subset Π^* of Π, and determine

$$\sup_{\pi \in \mathcal{V}} \inf_{\mathbb{P} \in \Pi^*} g(\pi; \mathbb{P}),$$

the robust maximal asymptotic growth rate (robust in Π^*). Moreover, we will find $\pi^* \in \mathcal{V}$ such that V^{π^*} attains (or surpasses) the maximal growth rate no matter which $\mathbb{P} \in \Pi^*$ materializes. This generalizes [73, Theorem 2.1] to the case with covariance uncertainty.

3.3   The Min-Max Result

In this section, we will first introduce generalized versions of the principal eigenvalue for the linear operator $L^{c(\cdot)}$ and a fully nonlinear operator F defined below. Then, we will investigate the relation between them on smooth bounded domains, and eventually extend the result to the entire domain E. The main technical result we obtain is Theorem 3.3.12. Finally, by using Theorem 3.3.12, we are able to resolve in Theorem 3.3.14 the problem proposed in Subsection 3.2.3.

Let us first recall the definition of Pucci's extremal operators. Given $0 < \lambda \leq \Lambda$,

we define for any $M \in \mathbb{S}^d$ the following matrix operators

$$\mathcal{M}^+_{\lambda,\Lambda}(M) := \Lambda \sum_{e_i(M)>0} e_i(M) + \lambda \sum_{e_i(M)<0} e_i(M),$$

(3.3.1)

$$\mathcal{M}^-_{\lambda,\Lambda}(M) := \lambda \sum_{e_i(M)>0} e_i(M) + \Lambda \sum_{e_i(M)<0} e_i(M).$$

From [28, p.15], we see that these operators can be expressed as

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A \in \mathcal{A}(\lambda,\Lambda)} \mathrm{Tr}(AM), \qquad \mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A \in \mathcal{A}(\lambda,\Lambda)} \mathrm{Tr}(AM),$$

where $\mathcal{A}(a,b)$ denotes the set of matrices in $\mathbb{S}^d$ with eigenvalues lying in $[a,b]$ for some real numbers $a \le b$. For general properties of Pucci's extremal operators, see e.g. [96] and [28, Section 2.2]. Now, let us define the operator $F : E \times \mathbb{S}^d \mapsto \mathbb{R}$ by

(3.3.2) $$F(x,M) := \frac{1}{2} \mathcal{M}^+_{\theta(x),\Theta(x)}(M) = \frac{1}{2} \sup_{A \in \mathcal{A}(\theta(x),\Theta(x))} \mathrm{Tr}(AM).$$

Let D be an open connected subset of E. Fixing $c \in \mathcal{C}$, we consider, for any given $\lambda \in \mathbb{R}$, the cone of positive harmonic functions with respect to $L^{c(\cdot)} + \lambda$ as

(3.3.3) $$H^c_\lambda(D) := \{\eta \in C^2(D) \mid L^{c(\cdot)}\eta + \lambda\eta = 0 \text{ and } \eta > 0 \text{ in } D\},$$

and set

(3.3.4) $$\lambda^{*,c}(D) := \sup\{\lambda \in \mathbb{R} \mid H^c_\lambda(D) \neq \emptyset\}.$$

Note that if D is a smooth bounded domain, $\lambda^{*,c}(D)$ coincides with the principal eigenvalue for $L^{c(\cdot)}$ on D; see e.g. [92, Theorem 4.3.2]. In our case, since we do not require the boundedness of D, $\lambda^{*,c}(D)$ is a generalized version of the principal eigenvalue for $L^{c(\cdot)}$ on D, which is also used in [74]. On the other hand, for any $\lambda \in \mathbb{R}$, we define

(3.3.5) $$H_\lambda(D) := \{\eta \in C^2(D) \mid F(x, D^2\eta) + \lambda\eta \le 0 \text{ and } \eta > 0 \text{ in } D\},$$

and set

(3.3.6)
$$\lambda^*(D) := \sup\{\lambda \in \mathbb{R} \mid H_\lambda(D) \neq \emptyset\},$$

which is a generalized version of the principal eigenvalue for the fully nonlinear operator F on D. For auxiliary purposes, we also consider, for any $\lambda \in \mathbb{R}$, the set

(3.3.7)
$$H_\lambda^+(D) := \{\eta \in C(\bar{D}) \mid F(x, D^2\eta) + \lambda\eta \leq 0 \text{ and } \eta > 0 \text{ in } D\},$$

where the inequality holds in viscosity sense. From this, we define

(3.3.8)
$$\lambda^+(D) := \sup\{\lambda \in \mathbb{R} \mid H_\lambda^+(D) \neq \emptyset\}.$$

For the special case where D is a smooth bounded domain, $\lambda^+(D)$ is the principal half-eigenvalue of the operator F on D that corresponds to positive eigenfunctions; see e.g. [97].

Lemma 3.3.1. *Given a smooth bounded domain $D \subset E$, there exists $\eta_D \in C(\bar{D})$ such that $\eta_D > 0$ in D and satisfies in viscosity sense the equation*

(3.3.9)
$$\begin{cases} F(x, D^2\eta_D) + \lambda^+(D)\eta_D = 0 & \text{in } D, \\ \eta_D = 0 & \text{on } \partial D. \end{cases}$$

Moreover, for any pair $(\lambda, \eta) \in \mathbb{R} \times C(\bar{D})$ with $\eta > 0$ in D which solves

(3.3.10)
$$\begin{cases} F(x, D^2\eta) + \lambda\eta = 0 & \text{in } D, \\ \eta = 0 & \text{on } \partial D, \end{cases}$$

(λ, η) must be of the form $(\lambda^+(D), \mu\eta_D)$ for some $\mu > 0$.

Proof. Let us introduce some properties of F. By definition, we see that

(3.3.11)
$$F(x, \mu M) = \mu F(x, M), \text{ for any } x \in E \text{ and } \mu \geq 0;$$

(3.3.12)
$$F \text{ is convex in } M.$$

Also, by [28, Lemma 2.10 (5)], for any $x \in E$ and $M, N \in \mathbb{S}^d$, we have

(3.3.13) $\qquad \frac{1}{2} \mathcal{M}^-_{\theta(x),\Theta(x)}(M - N) \leq F(x, M) - F(x, N) \leq \frac{1}{2} \mathcal{M}^+_{\theta(x),\Theta(x)}(M - N).$

Finally, we observe from (3.3.1) that F can be expressed as

$$F(x, M) = \frac{1}{2} \mathcal{M}^+_{\theta(x),\Theta(x)}(M) = \frac{1}{2} \left\{ \Theta(x) \sum_{e_i(M)>0} e_i(M) + \theta(x) \sum_{e_i(M)<0} e_i(M) \right\}.$$

From the continuity of θ and Θ in x, and the continuity of $e_i(M)$ in M for each i (see e.g. [84, p.497]), we conclude that

(3.3.14) $\qquad\qquad\qquad F$ is continuous in $E \times \mathbb{S}^d$.

Now, thanks to (3.3.11)-(3.3.14) and [97, Lemma 1.1], this lemma follows from [97, Theorems 1.1, 1.2]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.3.1 Regularity of η_D

In this subsection, we will show that, for any smooth bounded domain $D \subset E$, the continuous viscosity solution η_D given in Lemma 3.3.1 is actually smooth up to the boundary ∂D.

Let us consider the operator $J : \bar{D} \times \mathbb{S}^d \mapsto \mathbb{R}$ defined by

$$J(x, M) := F(x, M) + \lambda^+(D)\eta_D(x).$$

Lemma 3.3.2. η_D *belongs to* $C^{0,\beta}(\bar{D})$*, for any* $\beta \in (0, 1)$*.*

Proof. For any $x \in \bar{D}$ and $M, N \in \mathbb{S}^d$ with $M \geq N$, we deduce from (3.3.13) and (3.3.1) that

$$\begin{aligned}
(3.3.15) \qquad \frac{\theta_D}{2}\mathrm{Tr}(M - N) &\leq \frac{\theta(x)}{2}\mathrm{Tr}(M - N) = \frac{1}{2}\mathcal{M}^-_{\theta(x),\Theta(x)}(M - N) \\
&\leq F(x, M) - F(x, N) \leq \frac{1}{2}\mathcal{M}^+_{\theta(x),\Theta(x)}(M - N) \\
&= \frac{\Theta(x)}{2}\mathrm{Tr}(M - N) \leq \frac{\Theta_D}{2}\mathrm{Tr}(M - N),
\end{aligned}$$

where $\theta_D := \min_{x \in \bar{D}} \theta(x)$ and $\Theta_D := \max_{x \in \bar{D}} \Theta(x)$. On the other hand, recall that under Assumption III.1, $\theta, \Theta \in C^{0,\alpha}(\bar{D})$. Let K be a Hölder constant for both θ and Θ on $\bar{D}$. By (3.3.2) and (3.3.1), for any $x, y \in \bar{D}$ and $M \in \mathbb{S}^d$,

$$
\begin{aligned}
&|F(x, M) - F(y, M)| \\
(3.3.16) \quad &\leq \frac{1}{2}\left\{ |\Theta(x) - \Theta(y)| \sum_{e_i(M) > 0} e_i(M) + |\theta(x) - \theta(y)| \sum_{e_i(M) < 0} |e_i(M)| \right\} \\
&\leq Kd\|M\|\|x - y\|^\alpha.
\end{aligned}
$$

Under (3.3.11), (3.3.15), and (3.3.16), [17, Proposition 6] states that every bounded nonnegative viscosity solution to

$$
(3.3.17) \qquad\qquad J(x, D^2\eta) = 0 \;\; \text{in } D, \qquad \eta = 0 \;\; \text{on } \partial D
$$

is of the class $C^{0,\beta}(\bar{D})$ for all $\beta \in (0, 1)$. Thanks to Lemma 3.3.1, η_D is indeed a bounded nonnegative viscosity solution to the above equation, and thus the lemma follows. $\qquad\square$

Lemma 3.3.3. η_D *is the unique continuous viscosity solution to* (3.3.17).

Proof. By Lemma 3.3.1, we immediately have the viscosity solution property. To prove the uniqueness, it suffices to show that a comparison principle holds for $J(x, D^2\eta) = 0$. For any $x \in \bar{D}$ and $M, N \in \mathbb{S}^d$ with $M \geq N$, we see from the definition of J and (3.3.15) that

$$
(3.3.18) \qquad\qquad \frac{\theta_D}{2}\text{Tr}(M - N) \leq J(x, M) - J(x, N) \leq \frac{\Theta_D}{2}\text{Tr}(M - N).
$$

Thanks to this inequality, we conclude from [75, Theorem 2.6] that a comparison principle holds for $J(x, D^2\eta) = 0$. $\qquad\square$

The following regularity result is taken from [101, Theorem 1.2].

Lemma 3.3.4. *Suppose* $H : D \times \mathbb{S}^d \mapsto \mathbb{R}$ *satisfies the following conditions:*

(a) H is lower convex in $M \in \mathbb{S}^d$;

(b) there is a $\nu \in (0,1]$ s.t. $\nu|\xi|^2 \leq H(x, M + \xi\xi') - H(x, M) \leq \nu^{-1}|\xi|^2$ for all

$\xi \in \mathbb{R}^d$;

(c) there is a $K_1 > 0$ s.t. $|H(x,0)| \leq K_1$ for all $x \in D$;

(d) there are $K_2, K_3 > 0$ and $\beta \in (0,1)$ s.t. $\langle H(\cdot, M)\rangle_D^{(\beta)} \leq K_2 \sum_{i,j}|m_{ij}| + K_3$ for

all $M = \{m_{ij}\}_{i,j} \in \mathbb{S}^d$, where $\langle u\rangle_D^{(\beta)} := \sup_{x \in D, \rho > 0} \rho^{-\beta} \operatorname*{osc}_{D \cap B_\rho(x)} u$, for any $u : D \mapsto \mathbb{R}$.

Then the equation

$$H(x, D^2\eta) = 0 \quad in \ D, \qquad \eta = 0 \ on \ \partial D,$$

has a unique solution in the class $C^{2,\beta}(\bar{D})$ if $\beta \in (0, \bar\alpha)$, where the constant $\bar\alpha \in (0,1)$

depends only on d and ν.

Proposition 3.3.5. η_D belongs to $C^{2,\beta}(\bar{D})$ for any $\beta \in (0, \alpha \wedge \bar\alpha)$, where $\bar\alpha$ is given

in Lemma 3.3.4. This in particular implies $\lambda^+(D) = \lambda^*(D)$, and thus we have

(3.3.19)
$$\begin{cases} F(x, D^2\eta_D) + \lambda^*(D)\eta_D = 0 & in \ D, \\[2mm] \eta_D = 0 & on \ \partial D. \end{cases}$$

Proof. Let us show that the operator J satisfies conditions (a)-(d) in Lemma 3.3.4.

It is obvious from (3.3.12) that J satisfies (a). Since $\xi\xi' \geq 0$ and $\operatorname{Tr}(\xi\xi') = |\xi|^2$ for all

$\xi \in \mathbb{R}^d$, we see from (3.3.18) that J satisfies (b). By the continuity of η_D on $\bar{D}$, (c) is

also satisfied as $|J(x,0)| = 0 + \lambda^+(D)\eta_D(x) \leq K_1 := \lambda^+(D)\max_{\bar{D}} \eta_D$. To prove (d),

let us first observe that: for any $\beta \in (0,1)$ and $u \in C^{0,\beta}(D)$ with a Hölder constant

K, we have $\operatorname*{osc}_{D \cap B_\rho(x)} u \leq K\rho^\beta$, which yields $\langle u\rangle_D^{(\beta)} \leq K$. Recall that $\theta, \Theta \in C^{0,\alpha}(D)$

(Assumption III.1) and $\eta_D \in C^{0,\beta}(\bar{D})$ for all $\beta \in (0,1)$ (Lemma 3.3.2). Now, for

any $\beta \in (0, \alpha \wedge \bar\alpha)$, we have $\theta, \Theta, \eta_D \in C^{0,\beta}(D)$. Let K' be a Hölder constant for all

the three functions. Then, from the definition of J, the calculation (3.3.16), and the

fact that $\|M\| \leq d\|M\|_{\max} \leq d\sum_{i,j}|m_{ij}|$ for any $M = \{m_{ij}\}_{i,j} \in \mathbb{S}^d$, we conclude

that $J(\cdot, M) \in C^{0,\beta}(D)$ with a Hölder constant $d^2\left(\sum_{i,j}|m_{ij}|\right)K' + \lambda^+(D)K'$. It

follows that $\langle J(\cdot, M)\rangle_D^{(\beta)} \leq d^2\left(\sum_{i,j}|m_{ij}|\right)K' + \lambda^+(D)K'$. Thus, (d) is satisfied

for all $\beta \in (0, \alpha \wedge \bar{\alpha})$, with $K_2 := d^2 K'$ and $K_3 := \lambda^+(D)K'$. Now, we conclude

from Lemma 3.3.4 that there is a unique solution in $C^{2,\beta}(\bar{D})$ to (3.3.17) for all

$\beta \in (0, \alpha \wedge \bar{\alpha})$. However, in view of Lemma 3.3.3, this unique $C^{2,\beta}(\bar{D})$ solution can

only be η_D.

The fact that η_D is of the class $C^{2,\beta}(\bar{D})$ and solves (3.3.9) implies that $\lambda^+(D) \leq$

$\lambda^*(D)$. Since we have the opposite inequality just from the definitions of $\lambda^+(D)$ and

$\lambda^*(D)$, we conclude that $\lambda^+(D) = \lambda^*(D)$. Then (3.3.9) becomes (3.3.19). $\qquad\square$

3.3.2 Relation between $\lambda^*(D)$ and $\lambda^{*,c}(D)$

In this subsection, we will show that $\lambda^*(D) = \inf_{c \in \mathcal{C}} \lambda^{*,c}(D)$ for any smooth

bounded domain D.

Let us first state a maximum principle on small domains for the operator $G_\delta :$

$E \times \mathbb{R} \times \mathbb{S}^d \mapsto \mathbb{R}$ defined by

$$G_\delta(x, u, M) := -F(x, -M) - \delta|u| = \frac{1}{2}\mathcal{M}^-_{\theta(x),\Theta(x)}(M) - \delta|u|,$$

where δ can be any nonnegative real number.

Lemma 3.3.6. *For any smooth bounded domain $D \subset E$, there exists $\varepsilon_0 > 0$, depending on D, such that if a smooth bounded domain $U \subset D$ satisfies $Leb(U) < \varepsilon_0$, then every $\eta \in C(\bar{U})$ which is a viscosity solution to*

$$\begin{cases} G_\delta(x, \eta, D^2\eta) \leq 0 & in\ U, \\ \eta \geq 0 & on\ \partial U, \end{cases}$$

satisfies $\eta \geq 0$ in U.

Proof. Consider the operator $\bar{F} : E \times \mathbb{R} \times \mathbb{S}^d \mapsto \mathbb{R}$ defined by $\bar{F}(x, u, M) := F(x, M) + \delta|u|$. For any $x \in E$, $u, v \in \mathbb{R}$ and $M, N \in \mathbb{S}^d$, we see from (3.3.13) that

$$
(3.3.20) \quad
\begin{aligned}
\frac{1}{2}\mathcal{M}^-_{\theta(x),\Theta(x)}(M - N) - \delta|u - v| &\leq \bar{F}(x, u, M) - \bar{F}(x, v, N) \\
&\leq \frac{1}{2}\mathcal{M}^+_{\theta(x),\Theta(x)}(M - N) + \delta|u - v|.
\end{aligned}
$$

Moreover, by (3.3.14), we immediately have

$$
(3.3.21) \qquad \bar{F}(x, 0, M) = F(x, M) \text{ is continuous in } E \times \mathbb{S}^d.
$$

Noting that $G_\delta(x, u, M) = -\bar{F}(x, -u, -M)$, we have $G_\delta(x, u, M) - G_\delta(x, v, N) = \bar{F}(x, -v, -N) - \bar{F}(x, -u, -M)$. Then, by using (3.3.20), we get

$$
(3.3.22) \quad
\begin{aligned}
&G_\delta(x, u - v, M - N) \\
&= \frac{1}{2}\mathcal{M}^-_{\theta(x),\Theta(x)}(M - N) - \delta|u - v| \leq G_\delta(x, u, M) - G_\delta(x, v, N) \\
&\leq \frac{1}{2}\mathcal{M}^+_{\theta(x),\Theta(x)}(M - N) + \delta|u - v| = \bar{F}(x, u - v, M - N),
\end{aligned}
$$

which implies that the operator G_δ satisfies the (D_F) condition in [97, p.107] (with F replaced by $\bar{F}$). Now, thanks to (3.3.20)-(3.3.22), this lemma follows from [97, Theorem 3.5]. $\qquad\square$

Proposition 3.3.7. *For any smooth bounded domain $D \subset E$, $\lambda^*(D) \leq \inf_{c \in \mathcal{C}} \lambda^{*,c}(D)$.*

Proof. Assume the contrary that $\lambda^*(D) > \inf_{c \in \mathcal{C}} \lambda^{*,c}(D)$. Then there exists $\bar{c} \in \mathcal{C}$ such that $\lambda^*(D) > \lambda^{*,\bar{c}}(D)$. Take $\bar{\eta} \in C^2(D)$ with $\bar{\eta} > 0$ in D such that

$$
\begin{cases}
L^{\bar{c}(\cdot)}\bar{\eta} + \lambda^{*,\bar{c}}(D)\bar{\eta} = 0 & \text{in } D, \\
\bar{\eta} = 0 & \text{on } \partial D.
\end{cases}
$$

From the definition of F, we see that $\bar{\eta}$ is a viscosity subsolution to

$$
(3.3.23) \qquad F(x, D^2\eta) + \lambda^{*,\bar{c}}(D)\eta = 0 \text{ in } D.
$$

On the other hand, the function η_D, given in Lemma 3.3.1, is a viscosity supersolution to (3.3.23) as it solves (3.3.19) and $\lambda^*(D) > \lambda^{*,\bar{c}}(D)$. We claim that there exists $\ell > 0$ such that $\bar{\eta} \le \ell\eta_D$ in D. We will show this by following an argument used in the proof of [97, Theorem 4.1]. Take a compact subset K of D such that $Leb(D \setminus K) < \varepsilon_0$, where ε_0 is given in Lemma 3.3.6. By the continuity of $\bar{\eta}$ and η_D, there exists $\ell > 0$ such that $\ell\eta_D - \bar{\eta} > 0$ on K. Consider the function $f_\ell := \ell\eta_D - \bar{\eta}$. By (3.3.13) and (3.3.11),

$$G_{\lambda^{*,\bar{c}}(D)}(x, f_\ell, D^2 f_\ell) = -F(x, -D^2 f_\ell) - \lambda^{*,\bar{c}}(D)|f_\ell| \le -F(x, -D^2 f_\ell) + \lambda^{*,\bar{c}}(D)f_\ell$$

$$\le \ell F(x, D^2\eta_D) - F(x, D^2\bar{\eta}) + \lambda^{*,\bar{c}}(D)(\ell\eta_D - \bar{\eta}) \le 0 \text{ in } D,$$

where the last inequality follows from the supersolution property of η_D and the subsolution property of $\bar{\eta}$ to (3.3.23). Since $f_\ell \ge 0$ on $\partial(D \setminus K)$, we obtain from Lemma 3.3.6 that $f_\ell \ge 0$ on $D \setminus K$. Thus, we conclude that $\bar{\eta} \le \ell\eta_D$ in D. Now, by Perron's method we can construct a continuous viscosity solution v to (3.3.23) on D such that $\bar{\eta} \le v \le \ell\eta_D$. This in particular implies $v > 0$ in D and the pair $(\lambda^{*,\bar{c}}(D), v)$ solves (3.3.10). Recalling that $\lambda^+(D) = \lambda^*(D)$ from Proposition 3.3.5, we see that this is a contradiction to Lemma 3.3.1 as $\lambda^{*,\bar{c}}(D) < \lambda^*(D) = \lambda^+(D)$. $\square$

To prove the opposite inequality $\lambda^*(D) \ge \inf_{c\in\mathcal{C}} \lambda^{*,c}(D)$ for any smooth bounded domain $D \subset E$, we will make use of the theory of continuous selection pioneered by [85], and follow particularly the formulation in [25]. For a brief introduction to this theory and its adaptation to the current context, see Subsection C.

Proposition 3.3.8. *Let $D \subset E$ be a smooth bounded domain. If D is convex, then* $\lambda^*(D) \ge \inf_{c\in\mathcal{C}} \lambda^{*,c}(D)$.

Proof. We will construct a sequence $\{\bar{c}'_m\}_{m\in\mathbb{N}} \subset \mathcal{C}$ such that $\limsup_{m\to\infty} \lambda^{*,\bar{c}'_m}(D) \le \lambda^*(D)$, which gives the desired result.

Step 1: Constructing $\{\bar{c}'_m\}_{m\in\mathbb{N}}$. Recall that $\eta_D \in C^2(\bar{D})$ by Proposition 3.3.5.

Then, we deduce from (3.3.1) that there exists $\kappa > 0$ such that

(3.3.24)
$$\max\{|\lambda - \lambda'|, |\Lambda - \Lambda'|\} < \kappa$$
$$\Rightarrow \quad |\mathcal{M}^+_{\lambda,\Lambda}(D^2\eta_D(x)) - \mathcal{M}^+_{\lambda',\Lambda'}(D^2\eta_D(x))| < 2/m, \text{ for all } x \in \bar{D}.$$

Also, since $\|\cdot\|_{\max} \leq \|\cdot\|$, the map $(M, x) \mapsto L^M\eta_D(x)$ is continuous in M, uniformly in $x \in \bar{D}$. It follows that there exists $\beta > 0$ such that

(3.3.25)
$$\|N - M\| < \beta \quad \Rightarrow \quad |L^N\eta_D(x) - L^M\eta_D(x)| < 1/m \text{ for all } x \in \bar{D}.$$

Set $\xi := \min_{x\in\bar{D}}(\Theta - \theta)(x) > 0$ (recall that $\Theta > \theta$ in E under Assumption III.1). Now, by taking $\gamma := \theta + \frac{\kappa\wedge\xi}{4}$ and $\Gamma := \Theta - \frac{\kappa\wedge\xi}{4}$ in Proposition A.9, we obtain that there is a continuous function $c_m : \bar{D} \mapsto \mathbb{S}^d$ such that

(3.3.26) $\quad c_m(x) \in \mathcal{A}(\gamma(x), \Gamma(x)) \quad$ and $\quad F_{\gamma,\Gamma}(x, D^2\eta_D) \leq L^{c_m(\cdot)}\eta_D(x) + \dfrac{1}{m}, \quad \forall x \in \bar{D},$

where $F_{\gamma,\Gamma}(x, M)$ is defined in (A.0.2). By mollifying the function c_m, we can construct a function $\bar{c}_m : \bar{D} \mapsto \mathbb{S}^d$ such that $\bar{c}_m \in C^\infty(\bar{D})$ and $\|\bar{c}_m(x) - c_m(x)\|_{\max} < (\beta \wedge \frac{\kappa\wedge\xi}{4})/d$ for all $x \in \bar{D}$ (More precisely, $c_m \in C(\bar{D})$ implies that for any open set D' containing $\bar{D}$, there is a function $\tilde{c}_m \in C(D')$ such that $\tilde{c}_m = c_m$ on $\bar{D}$; see e.g. [53, Lemma 6.37]. Then by mollifying $\tilde{c}_m$, we get a sequence of smooth functions converging uniformly to $\tilde{c}_m$ on $\bar{D}$). It follows that

(3.3.27)
$$\|\bar{c}_m(x) - c_m(x)\| \leq d\|\bar{c}_m(x) - c_m(x)\|_{\max} < \beta \wedge \frac{\kappa \wedge \xi}{4} \text{ for all } x \in \bar{D}.$$

Combining (3.3.24)-(3.3.27), for each $x \in \bar{D}$, we see that $\bar{c}_m(x) \in \mathcal{A}(\theta(x), \Theta(x))$ and

(3.3.28)
$$F(x, D^2\eta_D) = \frac{1}{2}\mathcal{M}^+_{\theta(x),\Theta(x)}(D^2\eta_D(x)) < \frac{1}{2}\mathcal{M}^+_{\gamma(x),\Gamma(x)}(D^2\eta_D(x)) + \frac{1}{m}$$
$$= F_{\gamma,\Gamma}(x, D^2\eta_D) + \frac{1}{m} \leq L^{c_m(\cdot)}\eta_D(x) + \frac{2}{m} \leq L^{\bar{c}_m(\cdot)}\eta_D(x) + \frac{3}{m}.$$

Now, take some $\vec{c}'_m \in \mathcal{C}$ such that $\vec{c}'_m$ and $\bar{c}_m$ coincide on $\bar{D}$. Then (3.3.28) and the fact that $F(x, D^2\eta_D) + \lambda^*(D)\eta_D = 0$ in D (Proposition 3.3.5) imply

$$(3.3.29) \qquad |h_m| < 3/m \text{ in } D, \quad \text{where } h_m := L^{\vec{c}'_m(\cdot)}\eta_D + \lambda^*(D)\eta_D.$$

Step 2: Showing $\limsup_{m\to\infty} \lambda^{*,\bar{c}'_m}(D) \leq \lambda^*(D)$. In the following, we will use the argument in [52, Section 3 (starting from (3.3))]. Let η_m be the eigenfunction associated with the eigenvalue problem

$$\begin{cases} L^{\vec{c}'_m(\cdot)}\eta + \lambda^{*,\bar{c}'_m}(D)\eta = 0 & \text{in } D \\ \\ \eta = 0 & \text{on } \partial D. \end{cases}$$

Pick $x_0 \in D$. We define the normalized eigenfunction $\tilde{\eta}_m := \frac{\eta_D(x_0)}{\eta_m(x_0)}\eta_m$. By [95, Lemma on p.789], there exist $k_1, k_2 > 0$, independent of m, such that

$$(3.3.30) \qquad k_1 d(x, \partial D) \leq \tilde{\eta}_m(x) \leq k_2 d(x, \partial D), \quad \text{for all } x \in D.$$

Also, thanks to (3.3.11) and (3.3.15), we may apply [17, Proposition 1] and obtain some $\delta > 0$ and $C > 0$ such that $\eta_D(x) \leq Cd(x, \partial D)$ if $d(x, \partial D) < \delta$. Thus, we conclude that

$$(3.3.31) \qquad 1 \leq t_m := \sup_{x\in D} \frac{\eta_D(x)}{\tilde{\eta}_m(x)} < \infty.$$

Setting $s_m := t_m\lambda^*(D)/\lambda^{*,\bar{c}'_m}(D)$, we deduce from the definitions of t_m and s_m that

$$(3.3.32) \qquad L^{\vec{c}'_m(\cdot)}(s_m\tilde{\eta}_m - \eta_D) + h_m = -t_m\lambda^*(D)\tilde{\eta}_m + \lambda^*(D)\eta_D \leq 0 \text{ in } D.$$

Let w_m be the unique solution of the class $C^{2,\alpha}(D) \cap C(\bar{D})$ to the equation

$$(3.3.33) \qquad L^{\vec{c}'_m(\cdot)}w_m = h_m \quad \text{in } D, \qquad w_m = 0 \text{ on } \partial D.$$

Note that by [52, Remark 3.1], the convexity of D and (3.3.29) guarantee the existence of a constant $M > 0$, independent of m, such that

$$(3.3.34) \qquad |w_m(x)| \leq \frac{Md(x, \partial D)}{m}, \quad \text{for all } x \in D.$$

Combining (3.3.32) and (3.3.33), we get

$$\begin{cases} L^{\bar{c}'_m(\cdot)}(s_m\tilde{\eta}_m - \eta_D + w_m) \leq 0 & \text{in } D \\[3mm] s_m\tilde{\eta}_m - \eta_D + w_m = 0 & \text{on } \partial D. \end{cases}$$

We then conclude from the maximum principle that $s_m\tilde{\eta}_m - \eta_D + w_m \geq 0$ in D.

From the definition of s_m, this inequality gives

$$\frac{\lambda^*(D)}{\lambda^{*,\bar{c}'_m}(D)} \geq \frac{\eta_D(x)}{t_m\tilde{\eta}_m(x)} - \frac{w_m(x)}{t_m\tilde{\eta}_m(x)} \geq \frac{\eta_D(x)}{t_m\tilde{\eta}_m(x)} - \frac{M}{k_1 m} \quad \text{for all } x \in D,$$

where the last inequality follows from (3.3.34), (3.3.31), and (3.3.30). Now, take

a sequence $\{x_k\}_{k\in\mathbb{N}}$ in D such that $\frac{\eta_D(x_k)}{\eta_m(x_k)} \to t_m$. By plugging x_k into the above

inequality and taking limit in k, we get

$$\frac{\lambda^*(D)}{\lambda^{*,\bar{c}'_m}(D)} \geq 1 - \frac{M}{k_1 m},$$

which implies $\lambda^*(D) \geq \limsup_{m\to\infty} \lambda^{*,\bar{c}'_m}(D)$. $\qquad\square$

Combining Propositions 3.3.7 and 3.3.8, we have the following result:

Theorem 3.3.9. *Let $D \subset E$ be a smooth bounded domain. If D is convex, $\lambda^*(D) = \inf_{c\in\mathcal{C}} \lambda^{*,c}(D)$.*

3.3.3 Relation between $\lambda^*(E)$ and $\lambda^{*,c}(E)$

In this subsection, we will first characterize $\lambda^*(E)$ in terms of $\lambda^*(E_n)$, and then

generalize Theorem 3.3.9 from bounded domains to the entire space E.

Let us first consider some Harnack-type inequalities. Note that for any $D \subset \mathbb{R}^d$ and $p \in [1,\infty)$, we will denote by $\mathcal{L}^p(D)$ the space of measurable functions f satisfying $(\int_D |f(x)|^p dx)^{1/p} < \infty$.

Lemma 3.3.10. *Let $D \subset E$ be a smooth bounded domain. Let $H : E \times \mathbb{S}^d \mapsto \mathbb{R}$ be such that*

$$(3.3.35) \quad \exists\ 0 < \lambda \leq \Lambda\ s.t.\ \mathcal{M}^-_{\lambda,\Lambda}(M) \leq H(x, M) \leq \mathcal{M}^+_{\lambda,\Lambda}(M)\ \ \forall\ (x, M) \in D \times \mathbb{S}^d.$$

If $\{u_n\}_{n \in \mathbb{N}}$ is sequence of continuous nonnegative viscosity solutions to

$$(3.3.36) \qquad\qquad H(x, D^2 u_n) + \delta_n u_n = f_n \quad in\ D,$$

where $\{\delta_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $[0, \infty)$ and $f_n \in \mathcal{L}^d(D)$, then we have:

(i) for any compact set $K \subset D$, there is a constant $C > 0$, depending only on D, K, d, λ, Λ, $\sup_n \delta_n$, such that

$$(3.3.37) \qquad\qquad \sup_K u_n \leq C \left\{ \inf_K u_n + \|f_n\|_{\mathcal{L}^d(D)} \right\}.$$

(ii) Suppose H satisfies (3.3.11). Given $x_0 \in D$ and $R_0 > 0$ such that $B_{R_0}(x_0) \subset D$, there exists a constant $C > 0$, depending only on R_0, d, λ, Λ, $\sup_n \delta_n$, such that for any $0 < R < R_0$,

$$(3.3.38) \qquad \sup_{\bar{B}_R(x_0)} u_n \leq C \left\{ \inf_{\bar{B}_R(x_0)} u_n + R^2 \|f_n\|_{\mathcal{L}^d(B_{R_0}(x_0))} \right\}$$

As a consequence, if we assume further that $\{u_n\}_{n \in \mathbb{N}}$ is uniformly bounded and $\{f_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{L}^d(D)$, then for any compact connected set $K \subset D$ and $\beta \in (0, 1)$, $u_n \in C^{0,\beta}(K)$ for all $n \in \mathbb{N}$, with one fixed Hölder constant.

Proof. (i) Set $\delta^* := \sup_n \delta_n < \infty$. By (3.3.35), we have

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 u_n) + \delta^* u_n \geq H(x, D^2 u_n) + \delta_n u_n \geq \mathcal{M}^-_{\lambda,\Lambda}(D^2 u_n) - \delta^* u_n \quad in\ D.$$

In view of (3.3.36), we obtain $\mathcal{M}^+_{\lambda,\Lambda}(D^2 u_n) + \delta^* u_n \geq f_n \geq \mathcal{M}^-_{\lambda,\Lambda}(D^2 u_n) - \delta^* u_n$ in D. Thanks to this inequality, the estimate (3.3.37) follows from [97, Theorem 3.6].

(ii) Thanks to the estimate (3.3.37) and [53, Lemma 8.23], we can prove part (ii) by following the argument in the proof of [18, Corollary 3.2]. For a detailed proof, see Appendix B. □

Proposition 3.3.11. $\lambda^*(E) = \downarrow \lim_{n \to \infty} \lambda^*(E_n)$ *and there exists some* $\eta^* \in H_{\lambda^*(E)}(E)$ *such that*

$$(3.3.39) \qquad F(x, D^2\eta^*) + \lambda^*(E)\eta^* = 0 \quad in \ E.$$

Proof. It is obvious from the definition that $\lambda^*(E_n)$ is decreasing in n and $\lambda^*(E) \le \lambda^*(E_n)$ for all $n \in \mathbb{N}$. It follows that $\lambda^*(E) \le \lambda_0 := \downarrow \lim_{n \to \infty} \lambda^*(E_n)$. To prove the opposite inequality, it suffices to show that $H_{\lambda_0}(E) \ne \emptyset$. To this end, we take η_n as the eigenfunction given in Lemma 3.3.1 with $D = E_n$. Pick an arbitrary $x_0 \in E_1$, and define $\tilde{\eta}_n(x) := \frac{\eta_n(x)}{\eta_n(x_0)}$ such that $\tilde{\eta}_n(x_0) = 1$ for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. In view of Proposition 3.3.5, $\{\tilde{\eta}_m\}_{m>n}$ is a sequence of positive smooth solutions to

$$(3.3.40) \qquad F(x, D^2\tilde{\eta}_m) + \lambda^*(E_m)\tilde{\eta}_m = 0 \quad in \ E_{n+1}.$$

From the definition of F, we see that F satisfies (3.3.35) in E_n with $\lambda = \min_{x \in \bar{E}_n} \theta(x)$ and $\Lambda = \max_{x \in \bar{E}_n} \Theta(x)$. Thus, by Lemma 3.3.10 (i), there is a constant $C > 0$, independent of m, such that

$$\sup_{\bar{E}_n} \tilde{\eta}_m \le C \inf_{\bar{E}_n} \tilde{\eta}_m \le C,$$

which implies $\{\tilde{\eta}_m\}_{m>n}$ is uniformly bounded in $\bar{E}_n$. On the other hand, given $\beta \in (0, 1)$, Lemma 3.3.10 (ii) guarantees that $\tilde{\eta}_m \in C^{0,\beta}(\bar{E}_n)$ for all $m > n$, with a fixed Hölder constant. Therefore, by using the Arzela-Ascoli theorem, we conclude that $\tilde{\eta}_m$ converges uniformly, up to some subsequence, to some function η^* on $\bar{E}_n$.

Thanks to the stability result of viscosity solutions (see e.g. [48, Lemma II.6.2]), we obtain from (3.3.40) that η^* is a nonnegative continuous viscosity solution in E_n to

$$(3.3.41) \qquad\qquad F(x, D^2\eta^*) + \lambda_0\eta^* = 0.$$

Furthermore, since $\eta^*(x_0) = \lim_{m\to\infty} \eta_m(x_0) = 1$, we conclude from [17, Theorem 2], a strict maximum principle for eigenvalue problems of fully nonlinear operators, that $\eta^* > 0$ in E_n. Finally, noting that for any $\beta \in (0,1)$, $\eta^* \in C^{0,\beta}(\bar{E}_n)$ with its Hölder constant same as $\tilde{\eta}_m$'s, we may use Lemma 3.3.4, as in the proof of Proposition 3.3.5, to show that $\eta^* \in C^2(\bar{E}_n)$.

Since the results above hold for each $n \in \mathbb{N}$, we conclude that η^* belongs to $C^2(E)$, takes positive values in E, and satisfies (3.3.41) in E. It follows that $\eta^* \in H_{\lambda_0}(E)$, which yields $\lambda_0 \leq \lambda^*(E)$. Therefore, we get $\lambda^*(E) = \lambda_0$, and then (3.3.41) becomes (3.3.39). □

Now, we are ready to present the main technical result of this chapter.

Theorem 3.3.12. $\lambda^*(E) = \inf_{c\in\mathcal{C}} \lambda^{*,c}(E)$.

Proof. Thanks to [92, Theorem 4.4.1 (i)], Theorem 3.3.9, and Propositions 3.3.11,

$$\inf_{c\in\mathcal{C}} \lambda^{*,c}(E) = \inf_{c\in\mathcal{C}} \inf_{n\in\mathbb{N}} \lambda^{*,c}(E_n) = \inf_{n\in\mathbb{N}} \inf_{c\in\mathcal{C}} \lambda^{*,c}(E_n) = \inf_{n\in\mathbb{N}} \lambda^*(E_n) = \lambda^*(E).$$

□

Remark 3.3.13. For the special case where θ and Θ are merely two positive constants, the derivation of Theorem 3.3.12 can be much simpler. Since the operator $F(x, M) = \frac{1}{2}\mathcal{M}^+_{\theta,\Theta}(M)$ is now Pucci's operator with elliptic constants θ and Θ, we may apply [18, Theorem 3.5] and obtain a positive Hölder continuous viscosity solution η^* to

$$F(x, D^2\eta^*) + \bar{\lambda}(E)\eta^* = 0 \ \text{ in } E,$$

where $\bar{\lambda}(E) := \inf\{\lambda^+(D) \mid D \subset E \text{ is a smooth bounded domain}\}$. Then, Lemma 3.3.4 implies η^* is actually smooth, and thus $\bar{\lambda}(E) \leq \lambda^*(E)$. Since $\bar{\lambda}(E) \geq \lambda^*(E)$ by definition, we conclude that $\bar{\lambda}(E) = \lambda^*(E)$. Now, thanks to [92, Theorem 4.4.1 (i)] and the standard result $\lambda^+(E_n) = \inf_{c \in \mathcal{C}} \lambda^{*,c}(E_n)$ for Pucci's operator (see e.g. [27, Proposition 1.1 (ii)] and [95, Theorem I]), we get

$$\inf_{c \in \mathcal{C}} \lambda^{*,c}(E) = \inf_{c \in \mathcal{C}} \inf_{n \in \mathbb{N}} \lambda^{*,c}(E_n) = \inf_{n \in \mathbb{N}} \inf_{c \in \mathcal{C}} \lambda^{*,c}(E_n) = \inf_{n \in \mathbb{N}} \lambda^+(E_n) = \bar{\lambda}(E) = \lambda^*(E).$$

However, as pointed out in [74, Discussion], it is not reasonable for financial applications to assume that each $c \in \mathcal{C}$ is both continuous and uniformly elliptic in E. Therefore, we consider in this chapter the more general setting where θ and Θ are functions defined on E, which includes the case without uniform ellipticity.

3.3.4 Application

By Theorem 3.3.12 and mimicking the proof of [74, Theorem 2.1], we have the following result. Note that, for simplicity, we will write $\lambda^* = \lambda^*(E)$.

Theorem 3.3.14. *Take $\eta^* \in H_{\lambda^*}(E)$ and normalize it so that $\eta^*(x_0) = 1$. Define $\pi_t^* := e^{\lambda^* t} \nabla \eta^*(X_t)$ for all $t \geq 0$, and set*

$$\Pi^* := \left\{ \mathbb{P} \in \Pi \;\middle|\; \mathbb{P}\text{-}\liminf_{t \to \infty}(t^{-1} \log \eta^*(X_t)) \geq 0, \; \mathbb{P}\text{-}a.s. \right\}.$$

Then, we have $\pi^* \in \mathcal{V}$ and $g(\pi^*; \mathbb{P}) \geq \lambda^*$ for all $\mathbb{P} \in \Pi^*$. Moreover,

$$(3.3.42) \qquad\qquad \lambda^* = \sup_{\pi \in \mathcal{V}} \inf_{\mathbb{P} \in \Pi^*} g(\pi; \mathbb{P}) = \inf_{\mathbb{P} \in \Pi^*} \sup_{\pi \in \mathcal{V}} g(\pi; \mathbb{P}).$$

Proof. Set $V_t^* := V_t^{\pi^*} = 1 + \int_0^t e^{\lambda^* s} \nabla \eta^*(X_s)' dX_s$, $t \geq 0$. By applying Itô's rule to the process $e^{\lambda^* t} \eta^*(X_t)$ we see that $V_t^* \geq e^{\lambda^* t} \eta^*(X_t) > 0$ $\mathbb{P}$-a.s. for all $\mathbb{P} \in \Pi$. This already implies $\pi^* \in \mathcal{V}$. Also, by the construction of Π^*, we have $\mathbb{P}\text{-}\liminf_{t \to \infty}(t^{-1} \log(V_t^*)) \geq \lambda^*$ $\mathbb{P}$-a.s. for all $\mathbb{P} \in \Pi^*$. It follows that $g(\pi^*; \mathbb{P}) \geq \lambda^*$ for all $\mathbb{P} \in \Pi^*$, which in turn implies $\lambda^* \leq \sup_{\pi \in \mathcal{V}} \inf_{\mathbb{P} \in \Pi^*} g(\pi; \mathbb{P})$.

Now, for any $c \in \mathcal{C}$ and $n \in \mathbb{N}$, set $\lambda_n^{*,c} = \lambda^{*,c}(E_n)$, take $\eta_n^{*,c} \in H_{\lambda_n^{*,c}}^c(E_n)$ with $\eta_n^{*,c}(x_0) = 1$, and define the process $\widetilde{V}_n^c(t) := e^{\lambda_n^{*,c}t}\eta_n^{*,c}(X_t)$. Note that under any $\mathbb{P} \in \Pi$ such that $\mathbb{P} \ll_{loc} \mathbb{Q}^c$, we have $\widetilde{V}_n^c(t) = 1 + \int_0^t (\pi_n^{*,c})_s' dX_s$ with $(\pi_n^{*,c})_t := e^{\lambda_n^{*,c}t}\nabla\eta_n^{*,c}(X_t)$. This, however, may not be true for general $\mathbb{P} \in \Pi$. As shown in the proof of [74, Theorem 2.1], for any fixed $c \in \mathcal{C}$ and $n \in \mathbb{N}$, we have the following: **1.** there exists a solution $(\mathbb{P}_{x,n}^{*,c})_{x \in E_n}$ to the generalized martingale problem for the operator $L^{c(\cdot),\eta_n^{*,c}} := L^{c(\cdot)} + c\nabla \log \eta_n^{*,c} \cdot \nabla$; **2.** the coordinate process X under $(\mathbb{P}_{x,n}^{*,c})_{x \in E_n}$ is recurrent in E_n; **3.** $\mathbb{P}_{x,n}^{*,c} \ll_{loc} \mathbb{Q}^c$ (note that we conclude from the previous two conditions that $\mathbb{P}_{x,n}^{*,c} \in \Pi^*$); **4.** the process $V^\pi/\widetilde{V}_n^c$ is a nonnegative $\mathbb{P}_{x,n}^{*,c}$-supermartingale for all $\pi \in \mathcal{V}$. We therefore have the analogous result $g(\pi; \mathbb{P}_n^{*,c}) \leq g(\pi_n^{*,c}; \mathbb{P}_n^{*,c}) \leq \lambda_n^{*,c}$ for all $\pi \in \mathcal{V}$, which yields $\inf_{\mathbb{P}\in\Pi^*} \sup_{\pi\in\mathcal{V}} g(\pi; \mathbb{P}) \leq \lambda_n^{*,c}$. Now, thanks to [92, Theorem 4.4.1 (i)] and Theorem 3.3.12, we have

$$\inf_{\mathbb{P}\in\Pi^*} \sup_{\pi\in\mathcal{V}} g(\pi; \mathbb{P}) \leq \inf_{c\in\mathcal{C}} \lim_{n\to\infty} \lambda_n^{*,c} = \lambda^*.$$

$\square$

Remark 3.3.15. Note that the normalized eigenfunction η^* in the statement of Theorem 3.3.14 may not be unique. It follows that the set of measures Π^* and the min-max problem in (3.3.42) may differ with our choice of η^*. In spite of this, we would like to emphasize the following:

(i) No matter which η^* we choose, the robust maximal asymptotic growth rate λ^* stays the same.

(ii) At the first glance, it may seem restrictive to work with Π^*. However, by the same calculation in [74, Remark 2.2], we see that: no matter which η^* we choose, Π^* is large enough to contain all the probabilities in Π under which X is tight in E, and thus corresponds to those $\mathbb{P} \in \Pi$ such that X is stable.

CHAPTER IV

On the Multidimensional Controller-and-Stopper Games

4.1 Introduction

We consider a zero-sum stochastic differential game of control and stopping under a fixed time horizon $T > 0$. There are two players, the "controller" and the "stopper," and a state process X^α which can be manipulated by the controller through the selection of the control α. Suppose the game starts at time $t \in [0, T]$. While the stopper has the right to choose the duration of this game (in the form of a random time τ), she incurs the running cost $f(s, X_s^\alpha, \alpha_s)$ at every moment $t \le s < \tau$, and the terminal cost $g(X_\tau^\alpha)$ at the time the game stops. Given the instantaneous discount rate $c(s, X_s^\alpha)$, the stopper would like to minimize her expected discounted cost

$$(4.1.1) \qquad \mathbb{E}\left[\int_t^\tau e^{-\int_t^s c(u, X_u^\alpha)du} f(s, X_s^\alpha, \alpha_s)ds + e^{-\int_t^\tau c(u, X_u^\alpha)du} g(X_\tau^\alpha) \right]$$

over all choices of τ. At the same time, however, the controller plays against her by maximizing (4.1.1) over all choices of α.

Ever since the game of control and stopping was introduced by Maitra & Sudderth [82], it has been known to be closely related to some common problems in mathematical finance, such as pricing American contingent claims (see e.g. [64, 69, 70]) and minimizing the probability of lifetime ruin (see [16]). The game itself, however, has not been studied to a great extent except certain particular cases. Karatzas and Sud-

derth [68] study a zero-sum controller-and-stopper game in which the state process X^α is a one-dimensional diffusion along a given interval on $\mathbb{R}$. Under appropriate conditions they prove that this game has a value and describe fairly explicitly a saddle point of optimal choices. It turns out, however, difficult to extend their results to multidimensional cases, as their techniques rely heavily on theorems of optimal stopping for one-dimensional diffusions. To deal with zero-sum multidimensional games of control and stopping, Karatzas and Zamfirescu [71] develop a martingale approach; also see [10], [14] and [15]. Again, it is shown that the game has a value, and a saddle point of optimal choices is constructed. However, it is assumed to be that the controller can affect only the drift term of X^α.

There is yet another subtle discrepancy between the one-dimensional game in [68] and the multidimensional game in [71]: the use of "strategies". Typically, in a two-player game, the player who acts first would not choose a fixed static action. Instead, she prefers to employ a strategy, which will give different responses to different future actions the other player will take. This additional flexibility enables the player to further decrease (increase) the expected cost, if she is the minimizer (maximizer). For example, in a game with two controllers (see e.g. [38, 37, 49, 26, 23]), the controller who acts first employs a strategy, which is a function that takes the other controller's latter decision as input and generates a control. Note that the use of strategies is preserved in the one-dimensional controller-and-stopper game in [68]: what the stopper employs is not simply a stopping time, but a strategy in the form of a random time which depends on the controller's decision. This kind of dynamic interaction is missing, however, in the multidimensional case: in [71], the stopper is restricted to use stopping times, which give the same response to any choice the controller makes.

Multidimensional controller-and-stopper games are also covered in Hamadène &
Lepeltier [55] and Hamadène [54], as a special case of mixed games introduced there.
The main tool used in these papers is the theory of backward differential equations
with two reflecting barriers. Interestingly, even though the method in [55, 54] differs
largely from that in [71], these two papers also require a diffusion coefficient which
is not affected by the controller, and do not allow the use of strategies. This is
in contrast with the one-dimensional case in [68], where everything works out fine
without any of the above restrictions. It is therefore of interest to see whether we can
construct a new methodology under which multidimensional controller-and-stopper
games can be analyzed even when the conditions required in [71, 55, 54] fail to hold.

In this chapter, such a methodology is built, under a Markovian framework. On
the one hand, we allow both the drift and diffusion terms of the state process X^α to
be controlled. On the other hand, we allow the players to use strategies. Specifically,
we first define *non-anticipating* strategies in Definition 4.3.1. Then, in contrast to
two-controller games where both players use strategies, only the stopper chooses to
use strategies in our case (which coincides with the set-up in [68]). This is because
by the nature of a controller-and-stopper game, the controller cannot benefit from
using non-anticipating strategies; see Remark 4.3.5. With this observation in mind,
we give appropriate definitions of the upper value function U and the lower value
function V in (4.3.6) and (4.3.7) respectively. Under this set-up, one presumably
could construct a saddle point of optimal choices by imposing suitable assumptions
on the cost functions, the dynamics of X^α, the associated Hamiltonian, or the control
set (as is done in [68, 71, 54, 55]; see Remark 4.3.7). However, we have no plan to
impose assumptions for constructing a saddle point. Instead, we intend to work
under a rather general framework, and determine under what conditions the game

has a value (i.e. $U = V$) and how we can derive a PDE characterization for this value when it exists.

Our method is motivated by Bouchard & Touzi [24], where the weak dynamic programming principle for stochastic control problems was first introduced. By generalizing the weak dynamic programming principle in [24] to the context of controller-and-stopper games, we show that V is a viscosity supersolution and U^* is a viscosity subsolution to an obstacle problem for a Hamilton-Jacobi-Bellman equation, where U^* denotes the upper semicontinuous envelope of U defined as in (4.1.3). More specifically, we first prove a continuity result for an optimal stopping problem embedded in V (Lemma 4.4.1), which enables us to follow the arguments in [24, Theorem 3.5] even under the current context of controller-and-stopper games. We obtain, accordingly, a weak dynamic programming principle for V (Proposition 4.4.2), which is the key to proving the supersolution property of V (Propositions 4.4.5). On the other hand, by generalizing the arguments in Chapter 3 of Krylov [77], we derive a continuity result for an optimal control problem embedded in U (Lemma 4.5.5). This leads to a weak dynamic programming principle for U (Proposition 4.5.6), from which the subsolution property of U^* follows (Proposition 4.5.7). Finally, under appropriate conditions, we prove a comparison result for the associated obstacle problem. Since V is a viscosity supersolution and U^* is a viscosity subsolution, the comparison result implies $U^* \leq V$. Recalling that U^* is actually larger than V by definition, we conclude that $U^* = V$. This in particular implies $U = V$, i.e. the game has a value, and the value function is the unique viscosity solution to the associated obstacle problem. This is the main result of this chapter; see Theorem 4.6.3. Note that once we have this PDE characterization, we can compute the value of the game using a stochastic numerical scheme proposed in Bayraktar & Fahim [6].

Another important advantage of our method is that it does not require any non-degeneracy condition on the diffusion term of X^α. For the multidimensional case in [71, 55, 54], Girsanov's theorem plays a crucial role, which entails non-degeneracy of the diffusion term. Even for the one-dimensional case in [68], this non-degeneracy is needed to ensure the existence of the state process (in the weak sense). Note that Weerasinghe [109] actually follows the one-dimensional model in [68] and extends it to the case with degenerate diffusion term; but at the same time, she assumes boundedness of the diffusion term, and some specific conditions including twice differentiability of the drift term and concavity of the cost function.

It is worth noting that while [71, 55, 54] do not allow the use of strategies and require the diffusion coefficient be control-independent and non-degenerate, they allow for non-Markovian dynamics and cost structures, as well as for non-Lipschitz drift coefficients. As a first step to allowing the use of strategies and incorporating controlled, and possibly degenerate, diffusion coefficients in a zero-sum multidimensional controller-and-stopper game, this chapter focuses on proving the existence and characterization of the value of the game under a Markovian framework with Lipschitz coefficients. We leave the general non-Markovian and non-Lipschitz case for future research.

The structure of this chapter is as follows: in Section 4.2, we set up the framework of our study. In Section 4.3, we define strategies, and give appropriate definitions of the upper value function U and the lower value function V. In Sections 4.4 and 4.5, the supersolution property of V and the subsolution property U^* are derived, respectively. In Section 4.6, we prove a comparison theorem, which leads to the existence of the value of the game and the viscosity solution property of the value function.

4.1.1   Notation

We collect some notation and definitions here for readers' convenience.

- Given a probability space $(E, \mathcal{I}, P)$, we denote by $L^0(E, \mathcal{I})$ the set of real-valued random variables on $(E, \mathcal{I})$; for $p \in [1, \infty)$, let $L_n^p(E, \mathcal{I}, P)$ denote the set of $\mathbb{R}^n$-valued random variables R on $(E, \mathcal{I})$ s.t. $\mathbb{E}_P[|R|^p] < \infty$. For the "$n = 1$" case, we simply write L_1^p as L^p.

- $\mathbb{R}_+ := [0, \infty)$ and $\mathcal{S} := \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$.

- $\mathbb{M}^d$ denotes the set of $d \times d$ real matrices.

- Given $E \subseteq \mathbb{R}^n$, $\mathrm{LSC}(E)$ denotes the set of lower semicontinuous functions defined on E, and $\mathrm{USC}(E)$ denotes the set of upper semicontinuous functions defined on E.

- Let E be a normed space. For any $(t, x) \in [0, T] \times E$, we define two types of balls centered at (t, x) with radius $r > 0$ as follows

$$(4.1.2) \quad \begin{aligned} B_r(t, x) &:= \{(t', x') \in [0, T] \times E \mid |t' - t| < r, \ |x' - x| < r\}; \\ B(t, x; r) &:= \{(t', x') \in [0, T] \times E \mid t' \in (t - r, t], \ |x' - x| < r\}. \end{aligned}$$

We denote by $\bar{B}_r(t, x)$ and $\bar{B}(t, x, ; r)$ the closures of $B_r(t, x)$ and $B(t, x; r)$, respectively. Moreover, given $w : [0, T] \times E \mapsto \mathbb{R}$, we define the upper and lower semicontinuous envelopes of w, respectively, by

$$(4.1.3) \quad \begin{aligned} w^*(t, x) &:= \lim_{\delta \downarrow 0} \sup\{w(t', x') \mid (t', x') \in ([0, T) \times E) \cap B_\delta(t, x)\}; \\ w_*(t, x) &:= \lim_{\delta \downarrow 0} \inf\{w(t', x') \mid (t', x') \in ([0, T) \times E) \cap B_\delta(t, x)\}. \end{aligned}$$

4.2 Preliminaries

4.2.1 The Set-up

Fix $T > 0$ and $d \in \mathbb{N}$. For any $t \in [0, T]$, let $\Omega^t := C([t, T]; \mathbb{R}^d)$ be the canonical space of continuous paths equipped with the uniform norm $\|\tilde{\omega}\|_{t,T} := \sup_{s \in [t,T]} |\tilde{\omega}_s|$, $\tilde{\omega} \in \Omega^t$. Let W^t denote the canonical process on Ω^t, and $\mathbb{G}^t = \{\mathcal{G}^t_s\}_{s \in [t,T]}$ denote the natural filtration generated by W^t. Let $\mathbb{P}^t$ be the Wiener measure on $(\Omega^t, \mathcal{G}^t_T)$, and consider the collection of $\mathbb{P}^t$-null sets $\mathcal{N}^t := \{N \in \mathcal{G}^t_T \mid \mathbb{P}^t(N) = 0\}$ and its completion $\overline{\mathcal{N}}^t := \{A \subseteq \Omega^t \mid A \subseteq N \text{ for some } N \in \mathcal{N}^t\}$. Now, define $\overline{\mathbb{G}}^t = \{\overline{\mathcal{G}}^t_s\}_{s \in [t,T]}$ as the augmentation of $\mathbb{G}^t$ by the sets in $\overline{\mathcal{N}}^t$, i.e. $\overline{\mathcal{G}}^t_s := \sigma(\mathcal{G}^t_s \cup \overline{\mathcal{N}}^t)$, $s \in [t, T]$. For any $x \in \mathbb{R}^d$, we also consider $\mathcal{G}^{t,x}_s := \mathcal{G}^t_s \cap \{W^t_t = x\}$, $\forall s \in [t, T]$. For Ω^t, W^t, $\mathcal{N}^t$, $\overline{\mathcal{N}}^t$, $\mathcal{G}^t_s$, $\overline{\mathcal{G}}^t_s$ and $\mathcal{G}^{t,x}_s$, we drop the superscript t whenever $t = 0$.

Given $x \in \mathbb{R}^d$, we define for any $\tilde{\omega} \in \Omega^t$ the shifted path $(\tilde{\omega} + x)_\cdot := \tilde{\omega}_\cdot + x$, and for any $A \subseteq \Omega^t$ the shifted set $A + x := \{\tilde{\omega} \in \Omega^t \mid \tilde{\omega} - x \in A\}$. Then, we define the shifted Wiener measure $\mathbb{P}^{t,x}$ by $\mathbb{P}^{t,x}(F) := \mathbb{P}^t(F - x)$, $F \in \mathcal{G}^t_T$, and let $\overline{\mathbb{P}}^{t,x}$ denote the extension of $\mathbb{P}^{t,x}$ on $(\Omega^t, \overline{\mathcal{G}}^t_T)$. For $\mathbb{P}^{t,x}$ and $\overline{\mathbb{P}}^{t,x}$, we drop the superscripts t and x whenever $t = 0$ and $x = 0$. We let $\mathbb{E}$ denote the expectation taken under $\overline{\mathbb{P}}$.

Fix $t \in [0, T]$ and $\omega \in \Omega$. For any $\tilde{\omega} \in \Omega^t$, we define the concatenation of ω and $\tilde{\omega}$ at t as

$$(\omega \otimes_t \tilde{\omega})_r := \omega_r 1_{[0,t]}(r) + (\tilde{\omega}_r - \tilde{\omega}_t + \omega_t) 1_{(t,T]}(r), \ r \in [0, T].$$

Note that $\omega \otimes_t \tilde{\omega}$ lies in Ω. Consider the shift operator in space $\psi_t : \Omega^t \mapsto \Omega^t$ defined by $\psi_t(\tilde{\omega}) := \tilde{\omega} - \tilde{\omega}_t$, and the shift operator in time $\phi_t : \Omega \mapsto \Omega^t$ defined by $\phi_t(\omega) := \omega|_{[t,T]}$, the restriction of $\omega \in \Omega$ on $[t, T]$. For any $r \in [t, T]$, since ψ_t and ϕ_t are by definition continuous under the norms $\|\cdot\|_{t,r}$ and $\|\cdot\|_{0,r}$ respectively, $\psi_t : (\Omega^t, \mathcal{G}^t_r) \mapsto (\Omega^t, \mathcal{G}^t_r)$ and $\phi_t : (\Omega, \mathcal{G}_r) \mapsto (\Omega^t, \mathcal{G}^t_r)$ are Borel measurable. Then, for

any $\xi : \Omega \mapsto \mathbb{R}$, we define the shifted functions $\xi^{t,\omega} : \Omega \mapsto \mathbb{R}$ by

$$\xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \phi_t(\omega')) \text{ for } \omega' \in \Omega.$$

Given a random time $\tau : \Omega \mapsto [0, \infty]$, whenever $\omega \in \Omega$ is fixed, we simplify our notation as

$$\omega \otimes_\tau \tilde{\omega} = \omega \otimes_{\tau(\omega)} \tilde{\omega}, \quad \xi^{\tau,\omega} = \xi^{\tau(\omega),\omega}, \quad \phi_\tau = \phi_{\tau(\omega)}, \quad \psi_\tau = \psi_{\tau(\omega)}.$$

Definition 4.2.1. On the space Ω, we define, for each $t \in [0,T]$, the filtration $\mathbb{F}^t = \{\mathcal{F}_s^t\}_{s\in[0,T]}$ by

$$\mathcal{F}_s^t := \mathcal{J}_{s+}^t, \text{ where } \mathcal{J}_s^t := \begin{cases} \{\emptyset, \Omega\}, & \text{if } s \in [0,t], \\ \sigma\left(\phi_t^{-1}\psi_t^{-1}\mathcal{G}_s^{t,0} \cup \overline{\mathcal{N}}\right), & \text{if } s \in [t,T]. \end{cases}$$

We drop the superscript t whenever $t = 0$.

Remark 4.2.2. Given $t \in [0,T]$, note that $\mathcal{F}_s^t$ is a collection of subsets of Ω for each $s \in [0,T]$, whereas $\mathcal{G}_s^t$, $\overline{\mathcal{G}}_s^t$ and $\mathcal{G}_s^{t,x}$ are collections of subsets of Ω^t for each $s \in [t,T]$.

Remark 4.2.3. By definition, $\mathcal{J}_s = \overline{\mathcal{G}}_s \; \forall s \in [0,T]$; then the right continuity of $\overline{\mathbb{G}}$ implies $\mathcal{F}_s = \overline{\mathcal{G}}_s \; \forall s \in [0,T]$ i.e. $\mathbb{F} = \overline{\mathbb{G}}$. Moreover, from Lemma C.1 (iii) in Appendix C and the right continuity of $\overline{\mathbb{G}}$, we see that $\mathcal{F}_s^t \subseteq \overline{\mathcal{G}}_s = \mathcal{F}_s \; \forall s \in [0,T]$, i.e. $\mathbb{F}^t \subseteq \mathbb{F}$.

Remark 4.2.4. Intuitively, $\mathbb{F}^t$ represents the information structure one would have if one starts observing at time $t \in [0,T]$. More precisely, for any $s \in [t,T]$, $\mathcal{G}_s^{t,0}$ represents the information structure one obtains after making observations on W^t in the period $[t,s]$. One could then deduce from $\mathcal{G}_s^{t,0}$ the information structure $\phi_t^{-1}\psi_t^{-1}\mathcal{G}_s^{t,0}$ for W on the interval $[0,s]$.

We define $\mathcal{T}^t$ as the set of all $\mathbb{F}^t$-stopping times which take values in $[0, T]$ $\overline{\mathbb{P}}$-a.s., and $\mathcal{A}_t$ as the set of all $\mathbb{F}^t$-progressively measurable M-valued processes, where M is a separable metric space. Also, for any $\mathbb{F}$-stopping times τ_1, τ_2 with $\tau_1 \leq \tau_2$ $\overline{\mathbb{P}}$-a.s., we denote by $\mathcal{T}^t_{\tau_1, \tau_2}$ the set of all $\tau \in \mathcal{T}^t$ which take values in $[\tau_1, \tau_2]$ $\overline{\mathbb{P}}$-a.s. Again, we drop the sub- or superscript t whenever $t = 0$.

4.2.2 The State Process

Given $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}$, let $X^{t,x,\alpha}$ denote a $\mathbb{R}^d$-valued process satisfying the following SDE:

$$(4.2.1) \qquad dX_s^{t,x,\alpha} = b(s, X_s^{t,x,\alpha}, \alpha_s)ds + \sigma(s, X_s^{t,x,\alpha}, \alpha_s)dW_s, \quad s \in [t, T],$$

with the initial condition $X_t^{t,x,\alpha} = x$. Let $\mathbb{M}^d$ be the set of $d \times d$ real matrices. We assume that $b : [0, T] \times \mathbb{R}^d \times M \mapsto \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \times M \mapsto \mathbb{M}^d$ are deterministic Borel functions, and $b(t, x, u)$ and $\sigma(t, x, u)$ are continuous in (x, u); moreover, there exists $K > 0$ such that for any $t \in [0, T]$, $x, y \in \mathbb{R}^d$, and $u \in M$,

$$(4.2.2) \qquad |b(t, x, u) - b(t, y, u)| + |\sigma(t, x, u) - \sigma(t, y, u)| \leq K|x - y|,$$

$$(4.2.3) \qquad |b(t, x, u)| + |\sigma(t, x, u)| \leq K(1 + |x|).$$

The conditions above imply that: for any initial condition $(t, x) \in [0, T] \times \mathbb{R}^d$ and control $\alpha \in \mathcal{A}$, (4.2.1) admits a unique strong solution $X^{t,x,\alpha}_\cdot$. Moreover, without loss of generality, we define

$$(4.2.4) \qquad\qquad X_s^{t,x,\alpha} := x \quad \text{for } s < t.$$

Remark 4.2.5. Fix $\alpha \in \mathcal{A}$. Under (4.2.2) and (4.2.3), the same calculations in [90, Appendix] and [20, Proposition 1.2.1] yield the following estimates: for each $p \geq 1$,

there exists $C_p(\alpha) > 0$ such that for any $(t,x), (t',x') \in [0,T] \times \mathbb{R}^d$, and $h \in [0, T-t]$,

$$(4.2.5) \qquad \mathbb{E}\left[\sup_{0 \le s \le T} |X_s^{t,x,\alpha}|^p\right] \le C_p(1 + |x|^p);$$

$$(4.2.6) \qquad \mathbb{E}\left[\sup_{0 \le s \le t+h} |X_s^{t,x,\alpha} - x|^p\right] \le C_p h^{\frac{p}{2}}(1 + |x|^p);$$

$$(4.2.7) \qquad \mathbb{E}\left[\sup_{0 \le s \le T} |X_s^{t',x',\alpha} - X_s^{t,x,\alpha}|^p\right] \le C_p \left[|x' - x|^p + |t' - t|^{\frac{p}{2}}(1 + |x|^p)\right].$$

Remark 4.2.6 (flow property). By pathwise uniqueness of the solution to (4.2.1), for any $0 \le t \le s \le T$, $x \in \mathbb{R}^d$, and $\alpha \in \mathcal{A}$, we have the following two properties:

(i) $X_r^{t,x,\alpha}(\omega) = X_r^{s, X_s^{t,x,\alpha}, \alpha}(\omega) \ \forall \ r \in [s, T]$, for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$; see [20, Chapter 2] and [91, p.41].

(ii) By (1.16) in [49] and the discussion below it, for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$, we have

$$X_r^{t,x,\alpha}\left(\omega \otimes_s \phi_s(\omega')\right) = X_r^{s, X_s^{t,x,\alpha}(\omega), \alpha^{s,\omega}}(\omega') \ \ \forall r \in [s, T], \ \text{for} \ \overline{\mathbb{P}}\text{-a.e} \ \omega' \in \Omega;$$

see also [87, Lemma 3.3].

4.2.3 Properties of Shifted Objects

Let us first derive some properties of $\mathcal{F}_T^t$-measurable random variables.

Proposition 4.2.7. *Fix* $t \in [0,T]$ *and* $\xi \in L^0(\Omega, \mathcal{F}_T^t)$.

(i) $\mathcal{F}_T^t$ *and* $\mathcal{F}_t$ *are independent. In particular,* ξ *is independent of* $\mathcal{F}_t$.

(ii) *There exist* $\overline{N}, \overline{M} \in \overline{\mathcal{N}}$ *such that: for any fixed* $\omega \in \Omega \setminus \overline{N}$, $\xi^{t,\omega}(\omega') = \xi(\omega') \ \forall \omega' \in \Omega \setminus \overline{M}$.

Proof. See Appendix C.1. $\qquad \square$

Fix $\theta \in \mathcal{T}$. Given $\alpha \in \mathcal{A}$, we can define, for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$, a control $\alpha^{\theta,\omega} \in \mathcal{A}_{\theta(\omega)}$ by

$$\alpha^{\theta,\omega}(\omega') := \{\alpha_r^{\theta,\omega}(\omega')\}_{r \in [0,T]} = \{\alpha_r(\omega \otimes_\theta \phi_\theta(\omega'))\}_{r \in [0,T]}, \ \ \omega' \in \Omega;$$

see [24, proof of Proposition 5.4]. Here, we state a similar result for stopping times in $\mathcal{T}$.

Proposition 4.2.8. *Fix $\theta \in \mathcal{T}$. For any $\tau \in \mathcal{T}_{\theta,T}$, we have $\tau^{\theta,\omega} \in \mathcal{T}_{\theta(\omega),T}^{\theta(\omega)}$ for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$.*

Proof. See Appendix C.2. □

Let $\rho : M \times M \mapsto \mathbb{R}$ be any given metric on M. By [77, p.142], $\rho'(u,v) := \frac{2}{\pi} \arctan \rho(u,v) < 1$ for $u,v \in M$ is a metric equivalent to ρ, from which we can construct a metric on $\mathcal{A}$ by

$$(4.2.8) \qquad \tilde{\rho}(\alpha, \beta) := \mathbb{E}\left[\int_0^T \rho'(\alpha_t, \beta_t) dt \right] \text{ for } \alpha, \beta \in \mathcal{A}.$$

Now, we state a generalized version of Proposition 4.2.7 (ii) for controls $\alpha \in \mathcal{A}$.

Proposition 4.2.9. *Fix $t \in [0,T]$ and $\alpha \in \mathcal{A}_t$. There exists $\overline{N} \in \overline{\mathcal{N}}$ such that: for any $\omega \in \Omega \setminus \overline{N}$, $\tilde{\rho}(\alpha^{t,\omega}, \alpha) = 0$. Furthermore, for any $(s,x) \in [0,T] \times \mathbb{R}^d$, $X_r^{s,x,\alpha^{t,\omega}}(\omega') = X_r^{s,x,\alpha}(\omega')$, $r \in [s,T]$, for $\overline{\mathbb{P}}$-a.e. $\omega' \in \Omega$.*

Proof. See Appendix C.3. □

4.3 Problem Formulation

We consider a controller-and-stopper game under the finite time horizon $T > 0$. While the controller has the ability to affect the state process X^α through the selection of the control α, the stopper has the right to choose the duration of this game, in the form of a random time τ. Suppose the game starts at time $t \in [0,T]$. The stopper incurs the running cost $f(s, X_s^\alpha, \alpha_s)$ at every moment $t \le s < \tau$, and the terminal cost $g(X_\tau^\alpha)$ at the time the game stops, where f and g are some given deterministic functions. According to the instantaneous discount rate $c(s, X_s^\alpha)$ for

some given deterministic function c, the two players interact as follows: the stopper would like to stop optimally so that her expected discounted cost could be minimized, whereas the controller intends to act adversely against her by manipulating the state process X^α in a way that frustrates the effort of the stopper.

For any $t \in [0, T]$, there are two possible scenarios for this game. In the first scenario, the stopper acts first. At time t, while the stopper is allowed to use the information of the path of W up to time t for her decision making, the controller has advantage: she has access to not only the path of W up to t but also the stopper's decision. Choosing one single stopping time, as a result, might not be optimal for the stopper. Instead, she would like to employ a stopping *strategy* which will give different responses to different future actions the controller will take.

Definition 4.3.1. Given $t \in [0, T]$, we say a function $\pi : \mathcal{A} \mapsto \mathcal{T}_{t,T}$ is an admissible stopping strategy on the horizon $[t, T]$ if it satisfies the following conditions:

(i) for any $\alpha, \beta \in \mathcal{A}$, it holds for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$ that

$$(4.3.1) \qquad \text{if } \min\{\pi[\alpha](\omega), \pi[\beta](\omega)\} \leq \inf\left\{s \geq t \ \bigg| \ \int_t^s \rho'(\alpha_r(\omega), \beta_r(\omega))dr \neq 0\right\},$$
$$\text{then } \pi[\alpha](\omega) = \pi[\beta](\omega).$$

Recall that ρ' is a metric on M defined right above (4.2.8).

(ii) for any $s \in [0, t]$, if $\alpha \in \mathcal{A}_s$, then $\pi[\alpha] \in \mathcal{T}^s_{t,T}$.

(iii) for any $\alpha \in \mathcal{A}$ and $\theta \in \mathcal{T}$ with $\theta \leq t$, it holds for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$ that

$$\pi[\alpha]^{\theta,\omega}(\omega') = \pi[\alpha^{\theta,\omega}](\omega'), \text{ for } \overline{\mathbb{P}}\text{-a.e. } \omega' \in \Omega.$$

We denote by $\Pi_{t,T}$ the set of all admissible stopping strategies on the horizon $[t, T]$.

Remark 4.3.2. Definition 4.3.1 (i) serves as the non-anticipativity condition for the stopping strategies. The intuition behind it should be clear: Suppose we begin

our observation at time t, and employ a strategy $\pi \in \Pi_{t,T}$. By taking the control α and following the path ω, we decide to stop at the moment $\pi[\alpha](\omega)$. If, up to this moment, we actually cannot distinguish between the controls α and β , then we should stop at the same moment if we were taking the control β.

Moreover, as shown in Proposition 4.3.3 below, (4.3.1) is equivalent to:

(4.3.2)
$$\text{For any } \alpha, \beta \in \mathcal{A} \text{ and } s \in [t, T],$$

$$1_{\{\pi[\alpha] \leq s\}} = 1_{\{\pi[\beta] \leq s\}} \text{ for } \overline{\mathbb{P}}\text{-a.e. } \omega \in \{\alpha =_{[t,s)} \beta\},$$

where $\{\alpha =_{[t,s)} \beta\} := \{\omega \in \Omega \mid \alpha_r(\omega) = \beta_r(\omega) \text{ for a.e. } r \in [t, s)\}$. This shows that Definition 4.3.1 (i) extends the non-anticipativity of strategies from two-controller games (see e.g. [26]) to current context of controller-and-stopper games.

Also notice that (4.3.2) is similar to, yet a bit weaker than, Assumption (C5) in [23]. This is because in the definition of $\{\alpha =_{[t,s)} \beta\}$, [23] requires $\alpha_r = \beta_r$ for all, instead of almost every, $r \in [t, s)$.

Proposition 4.3.3. *Fix $t \in [0, T]$. For any function $\pi : \mathcal{A} \mapsto \mathcal{T}_{t,T}$, (4.3.1) holds iff (4.3.2) holds.*

Proof. For any $\alpha, \beta \in \mathcal{A}$, we set $\theta(\omega) := \inf\{s \geq t \mid \int_t^s \rho'(\alpha_r(\omega), \beta_r(\omega))dr \neq 0\}$.

Step 1: Suppose π satisfies (4.3.1). For any $\alpha, \beta \in \mathcal{A}$, take some $N \in \overline{\mathcal{N}}$ such that (4.3.1) holds for $\omega \in \Omega \setminus N$. Fix $s \in [t, T]$. Given $\omega \in \{\alpha =_{[t,s)} \beta\} \setminus N$, we have $s \leq \theta(\omega)$. If $\pi[\alpha](\omega) \leq \theta(\omega)$, then (4.3.1) implies $\pi[\alpha](\omega) = \pi[\beta](\omega)$, and thus $1_{\{\pi[\alpha] \leq s\}}(\omega) = 1_{\{\pi[\beta] \leq s\}}(\omega)$. If $\pi[\alpha](\omega) > \theta(\omega)$, then (4.3.1) implies $\pi[\beta](\omega) > \theta(\omega)$ too. It follows that $1_{\{\pi[\alpha] \leq s\}}(\omega) = 0 = 1_{\{\pi[\beta] \leq s\}}(\omega)$, since $s \leq \theta(\omega)$. This already proves (4.3.2).

Step 2: Suppose (4.3.2) holds. Fix $\alpha, \beta \in \mathcal{A}$. By (4.3.2), there exists some

$N \in \overline{\mathcal{N}}$ such that

(4.3.3)   for any $s \in \mathbb{Q} \cap [t, T]$, $1_{\{\pi[\alpha] \leq s\}} = 1_{\{\pi[\beta] \leq s\}}$ for $\omega \in \{\alpha =_{[t,s)} \beta\} \setminus N$.

Fix $\omega \in \Omega \setminus N$. For $s \in \mathbb{Q} \cap [t, \theta(\omega)]$, we have $\omega \in \{\alpha =_{[t,s)} \beta\}$. Then (4.3.3) yields

(4.3.4) $1_{\{\pi[\alpha] \leq s\}}(\omega) = 1_{\{\pi[\beta] \leq s\}}(\omega)$, for all $s \in \mathbb{Q} \cap [t, \theta(\omega)]$.

If $\pi[\alpha](\omega) \leq \theta(\omega)$, take an increasing sequence $\{s_n\}_{n \in \mathbb{N}} \subset \mathbb{Q} \cap [t, \theta(\omega)]$ such that $s_n \uparrow \pi[\alpha](\omega)$. Then (4.3.4) implies $\pi[\beta](\omega) > s_n$ for all n, and thus $\pi[\beta](\omega) \geq \pi[\alpha](\omega)$. Similarly, by taking a decreasing sequence $\{r_n\}_{n \in \mathbb{N}} \subset \mathbb{Q} \cap [t, \theta(\omega)]$ such that $r_n \downarrow \pi[\alpha](\omega)$, we see from (4.3.4) that $\pi[\beta] \leq r_n$ for all n, and thus $\pi[\beta](\omega) \leq \pi[\alpha](\omega)$. We therefore conclude $\pi[\beta](\omega) = \pi[\alpha](\omega)$. Now, if $\pi[\beta](\omega) \leq \theta(\omega)$, we may argue as above to show that $\pi[\alpha](\omega) = \pi[\beta](\omega)$. This proves (4.3.1).   □

Next, we give concrete examples of strategies under Definition 4.3.1.

Example 4.3.4. Given $t \in [0, T]$, define $\lambda_t : \Omega \mapsto \Omega$ by $(\lambda_t(\omega)). := \omega_{. \wedge t}$. Recall the space $C([t, T]; \mathbb{R}^d)$ of continuous functions mapping $[t, T]$ into $\mathbb{R}^d$. For any $x \in \mathbb{R}^d$, we define $\pi : \mathcal{A} \mapsto \mathcal{T}_{t,T}$ by

(4.3.5) $$\pi[\alpha](\omega) := S\left(\{X_r^{t,x,\alpha}(\omega)\}_{r \in [t,T]}\right),$$

for some function $S : C([t, T]; \mathbb{R}^d) \mapsto [t, T]$ satisfying $\{\xi \mid S(\xi) \leq s\} \in \lambda_s^{-1} \mathcal{X}_T^t$ $\forall s \in [t, T]$, where $\mathcal{X}_T^t$ denotes the Borel σ-algebra generated by $C([t, T]; \mathbb{R}^d)$. Note that the formulation (4.3.5) corresponds to the stopping rules introduced in the one-dimensional controller-and-stopper game in [68], and it covers concrete examples such as exit strategies of a Borel set (see e.g. (4.5.13) below). We claim that Definition 4.3.1 readily includes the formulation (4.3.5).

Let the function $\pi : \mathcal{A} \mapsto \mathcal{T}_{t,T}$ be given as in (4.3.5). First, for any $\alpha, \beta \in \mathcal{A}$, set $\theta := \inf\{s \geq t \mid \int_t^s \rho'(\alpha_r(\omega), \beta_r(\omega)) dr \neq 0\}$. Observing that the strong solutions

$X^{t,x,\alpha}$ and $X^{t,x,\beta}$ coincide on the interval $[t,\theta)$ $\overline{\mathbb{P}}$-a.s., we conclude that π satisfies Definition 4.3.1 (i). Next, for any $s \in [0,t]$, since $X^{t,x,\alpha}$ depends on $\mathcal{F}_s$ only through the control α, Definition 4.3.1 (ii) also holds for π. To check Definition 4.3.1 (iii), let us introduce, for any $\theta \in \mathcal{T}$ with $\{\theta \leq t\} \notin \overline{\mathcal{N}}$, the strong solution $\tilde{X}$ to the SDE (4.2.1) with the drift coefficient $\tilde{b}(s,x,u) := 1_{\{s<t\}}0 + 1_{\{s\geq t\}}b(s,x,u)$ and the diffusion coefficient $\tilde{\sigma}(s,x,u) := 1_{\{s<t\}}0 + 1_{\{s\geq t\}}\sigma(s,x,u)$. Then, by using the pathwise uniqueness of strong solutions and Remark 4.2.6 (ii), for $\overline{\mathbb{P}}$-a.e. $\omega \in \{\theta \leq t\}$,

$$X_r^{t,x,\alpha}(\omega \otimes_\theta \phi_\theta(\omega')) = \tilde{X}_r^{0,x,\alpha}(\omega \otimes_\theta \phi_\theta(\omega')) = \tilde{X}_r^{\theta(\omega),\tilde{X}_\theta^{0,x,\alpha}(\omega),\alpha^{\theta,\omega}}(\omega')$$

$$= \tilde{X}_r^{\theta(\omega),x,\alpha^{\theta,\omega}}(\omega') = X_r^{t,x,\alpha^{\theta,\omega}}(\omega'),$$

$\forall r \in [t,T]$, for $\overline{\mathbb{P}}$-a.e. $\omega' \in \Omega$. This implies

$$\pi[\alpha]^{\theta,\omega}(\omega') = S(\{X_r^{t,x,\alpha}(\omega \otimes_\theta \phi_\theta(\omega'))\}_{r\in[t,T]}) = S(\{X_r^{t,x,\alpha^{\theta,\omega}}(\omega')\}_{r\in[t,T]}) = \pi[\alpha^{\theta,\omega}](\omega'),$$

for $\overline{\mathbb{P}}$-a.e. $\omega' \in \Omega$, which is Definition 4.3.1 (iii).

Let us now look at the second scenario in which the controller acts first. In this case, the stopper has access to not only the path of W up to time t but also the controller's decision. The controller, however, does not use strategies as an attempt to offset the advantage held by the stopper. As the next remark explains, the controller merely chooses one single control because she would not benefit from using non-anticipating strategies.

Remark 4.3.5. Fix $t \in [0,T]$. Let $\gamma : \mathcal{T} \mapsto \mathcal{A}_t$ satisfy the following non-anticipativity condition: for any $\tau_1, \tau_2 \in \mathcal{T}$ and $s \in [t,T]$, it holds for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$ that

$$\text{if } \min\{\tau_1(\omega), \tau_2(\omega)\} > s, \text{ then } (\gamma[\tau_1])_r(\omega) = (\gamma[\tau_2])_r(\omega) \text{ for } r \in [t,s].$$

Then, observe that $\gamma[\tau](\omega) = \gamma[T](\omega)$ on $[t,\tau(\omega))$ $\overline{\mathbb{P}}$-a.s. for any $\tau \in \mathcal{T}$. This implies

that employing the strategy γ has the same effect as employing the control $\gamma[T]$. In other words, the controller would not benefit from using non-anticipating strategies.

Now, we are ready to introduce the upper and lower value functions of the game of control and stopping. For $(t,x) \in [0,T] \times \mathbb{R}^d$, if the stopper acts first, the associated value function is

$$
U(t,x) := \inf_{\pi \in \Pi_{t,T}} \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}\left[\int_t^{\pi[\alpha]} e^{-\int_t^s c(u,X_u^{t,x,\alpha})du} f(s,X_s^{t,x,\alpha},\alpha_s)ds \right.
$$
$$
\left. + e^{-\int_t^{\pi[\alpha]} c(u,X_u^{t,x,\alpha})du} g(X_{\pi[\alpha]}^{t,x,\alpha}) \right].
$$
(4.3.6)

On the other hand, if the controller acts first, the associated value function is

$$
V(t,x) := \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} \mathbb{E}\left[\int_t^\tau e^{-\int_t^s c(u,X_u^{t,x,\alpha})du} f(s,X_s^{t,x,\alpha},\alpha_s)ds \right.
$$
$$
\left. + e^{-\int_t^\tau c(u,X_u^{t,x,\alpha})du} g(X_\tau^{t,x,\alpha}) \right].
$$
(4.3.7)

By definition, we have $U \geq V$. We therefore call U the upper value function, and V the lower value function. We say the game has a value if these two functions coincide.

Remark 4.3.6. In a game with two controllers (see e.g. [38, 37, 49, 26]), upper and lower value functions are also introduced. However, since both of the controllers use strategies, it is difficult to tell, just from the definitions, whether one of the value functions is larger than the other (despite their names). In contrast, in a controller-stopper game, only the stopper uses strategies, thanks to Remark 4.3.5. We therefore get $U \geq V$ for free, which turns out to be a crucial relation in the PDE characterization for the value of the game.

We assume that the cost functions f,g and the discount rate c satisfy the following conditions: $f : [0,T] \times \mathbb{R}^d \times M \mapsto \mathbb{R}_+$ is Borel measurable, and $f(t,x,u)$ is continuous in (x,u), and continuous in x uniformly in $u \in M$ for each t; $g : \mathbb{R}^d \mapsto \mathbb{R}_+$ is continuous; $c : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}_+$ is continuous and bounded above by some real

number $\bar{c} > 0$. Moreover, we impose the following polynomial growth condition on f and g

(4.3.8) $$|f(t,x,u)| + |g(x)| \leq K(1 + |x|^{\bar{p}}) \text{ for some } \bar{p} \geq 1.$$

Remark 4.3.7. Presumably, by imposing additional assumptions, one could construct a saddle point of optimal choices for a controller-and-stopper game. For example, in the one-dimensional game in [68], a saddle point is constructed under additional assumptions on the cost function and the dynamics of the state process (see (6.1)-(6.3) in [68]). For the multidimensional case, in order to find a saddle point, [71] assumes that the cost function and the drift coefficient are continuous with respect to the control variable, and the associated Hamiltonian always attains its infimum (see (71)-(73) in [71]); whereas [54] and [55] require compactness of the control set.

In this chapter, we have no plan to impose additional assumptions for constructing saddle points. Instead, we intend to investigate, under a rather general set-up, whether the game has a value and how we can characterize this value if it exists.

Remark 4.3.8. For any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}$, the polynomial growth condition (4.3.8) and (4.2.5) imply that

(4.3.9)
$$\mathbb{E}\left[\sup_{t \leq r \leq T}\left(\int_t^r e^{-\int_t^s c(u,X_u^{t,x,\alpha})du} f(s,X_s^{t,x,\alpha},\alpha_s)ds + e^{-\int_t^r c(u,X_u^{t,x,\alpha})du} g(X_r^{t,x,\alpha})\right)\right] < \infty.$$

Lemma 4.3.9. *Fix $\alpha \in \mathcal{A}$ and $(s,x) \in [0,T] \times \mathbb{R}^d$. For any $\{(s_n,x_n)\}_{n \in \mathbb{N}} \subset [0,T] \times \mathbb{R}^d$ such that $(s_n,x_n) \to (s,x)$, we have*

(4.3.10) $$\mathbb{E}\left[\sup_{0 \leq r \leq T} |g(X_r^{s_n,x_n,\alpha}) - g(X_r^{s,x,\alpha})|\right] \to 0;$$

(4.3.11) $$\mathbb{E}\int_0^T |1_{[s_n,T]}(r)f(r,X_r^{s_n,x_n,\alpha},\alpha_r) - 1_{[s,T]}(r)f(r,X_r^{s,x,\alpha},\alpha_r)|dr \to 0.$$

Proof. In view of (4.2.7), we have, for any $p \geq 1$,

$$(4.3.12) \qquad \mathbb{E}\left[\sup_{0 \leq r \leq T} |X_r^{s_n, x_n, \alpha} - X_r^{s, x, \alpha}|^p\right] \to 0.$$

Thanks to the above convergence and the polynomial growth condition (4.3.8) on f, we observe that (4.3.11) is a consequence of [77, Lemma 2.7.6].

It remains to prove (4.3.10). Fix $\varepsilon, \eta > 0$. Take $a > 0$ large enough such that $\frac{2C_1 T(2 + |x|)}{a} < \frac{\eta}{3}$, where $C_1 > 0$ is given as in Remark 4.2.5. Since g is continuous, it is uniformly continuous on $\bar{B}_a(x) := \{y \in \mathbb{R}^d \mid |y - x| \leq a\}$. Thus, there exists some $\delta > 0$ such that $|g(x) - g(y)| < \varepsilon$ for all $x, y \in \bar{B}_a(x)$ with $|x - y| < \delta$. Define

$$A := \left\{\sup_{0 \leq r \leq T} |X_r^{s, x, \alpha} - x| > a\right\}, \quad B_n := \left\{\sup_{0 \leq r \leq T} |X_r^{s_n, x_n, \alpha} - x| > a\right\},$$

$$B_n' := \left\{\sup_{0 \leq r \leq T} |X_r^{s_n, x_n, \alpha} - x_n| > \frac{a}{2}\right\}, \quad D_n := \left\{\sup_{0 \leq r \leq T} |X_r^{s_n, x_n, \alpha} - X_r^{s, x, \alpha}| \geq \delta\right\}.$$

By the Markov inequality and (4.2.6),

$$\bar{\mathbb{P}}(A) \leq \frac{C_1 \sqrt{T}(1 + |x|)}{a} < \frac{\eta}{3}, \quad \bar{\mathbb{P}}(B_n') \leq \frac{2C_1 \sqrt{T}(1 + |x_n|)}{a} < \frac{\eta}{3} \quad \text{for } n \text{ large enough}.$$

On the other hand, (4.3.12) implies that $\bar{\mathbb{P}}(D_n) < \frac{\eta}{3}$ for n large enough. Noting that $(B_n')^c \subseteq B_n^c$ for n large enough, we obtain

$$\mathbb{P}\left(\sup_{0 \leq r \leq T} |g(X_r^{s_n, x_n, \alpha}) - g(X_r^{s, x, \alpha})| > \varepsilon\right) \leq 1 - \bar{\mathbb{P}}(A^c \cap B_n^c \cap D_n^c) = \bar{\mathbb{P}}(A \cup B_n \cup D_n)$$

$$\leq \bar{\mathbb{P}}(A \cup B_n' \cup D_n) < \eta, \text{ for } n \text{ large enough}.$$

Thus, we have $h_n := \sup_{0 \leq r \leq T} |g(X_r^{s_n, x_n, \alpha}) - g(X_r^{s, x, \alpha})| \to 0$ in probability. Finally, observing that the polynomial growth condition (4.3.8) on g and (4.2.5) imply that $\{h_n\}_{n \in \mathbb{N}}$ is L^2-bounded, we conclude that $h_n \to 0$ in L^1, which gives (4.3.10). $\square$

4.3.1 The Associated Hamiltonian

For $(t, x, p, A) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{M}^d$, we associate the following Hamiltonian with our mixed control/stopping problem:

$$(4.3.13) \qquad H(t, x, p, A) := \inf_{a \in M} H^a(t, x, p, A),$$

where

$$H^a(t, x, p, A) := -b(t, x, a) \cdot p - \frac{1}{2} Tr[\sigma\sigma'(t, x, a)A] - f(t, x, a).$$

Since b, σ, and f are assumed to be continuous only in (x, a), and M is a separable metric space without any compactness assumption, the operator H may be neither upper nor lower semicontinuous. As a result, we will need to consider an upper semicontinuous version of H defined by

$$(4.3.14) \qquad \overline{H}(t, x, p, A) := \inf_{a \in M} (H^a)^*(t, x, p, A),$$

where $(H^a)^*$ is the upper semicontinuous envelope of H^a, defined as in (4.1.3); see Proposition 4.4.5. On the other hand, we will need to consider the lower semicontinuous envelope H_*, defined as in (4.1.3), in Proposition 4.5.7. Notice that $\overline{H}$ is different from the upper semicontinuous envelope H^*, defined as in (4.1.3) (in fact, $\overline{H} \geq H^*$). See Remark 4.4.7 for our choice of $\overline{H}$ over H^*.

4.3.2 Reduction to the Mayer Form

Given $t \in [0, T]$ and $\alpha \in \mathcal{A}_t$, let us increase the state process to (X, Y, Z), where

$$dY_s^{t,x,y,\alpha} = -Y_s^{t,x,y,\alpha} c(s, X_s^{t,x,\alpha}) ds, \ \ s \in [t, T], \ \text{with } Y_t^{t,x,y,\alpha} = y \geq 0;$$

$$Z_s^{t,x,y,z,\alpha} := z + \int_t^s Y_r^{t,x,y,\alpha} f(r, X_r^{t,x,\alpha}, \alpha_r) dr, \ \ \text{for some } z \geq 0.$$

Set $\mathcal{S} := \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$. For any $\mathbf{x} := (x, y, z) \in \mathcal{S}$, we define

$$\mathbf{X}_s^{t,\mathbf{x},\alpha} := \begin{pmatrix} X_s^{t,x,\alpha} \\ Y_s^{t,x,y,\alpha} \\ Z_s^{t,x,y,z,\alpha} \end{pmatrix},$$

and consider the function $F : \mathcal{S} \mapsto \mathbb{R}_+$ defined by

$$F(x, y, z) := z + yg(x).$$

Now, we introduce the functions $\bar{U}, \bar{V} : [0, T] \times \mathcal{S} \mapsto \mathbb{R}$ defined by

$$\bar{U}(t, x, y, z) := \inf_{\pi \in \Pi_{t,T}} \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}\left[F(X_{\pi[\alpha]}^{t,x,\alpha}, Y_{\pi[\alpha]}^{t,x,y,\alpha}, Z_{\pi[\alpha]}^{t,x,y,z,\alpha})\right] = \inf_{\pi \in \Pi_{t,T}} \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}\left[F(\mathbf{X}_{\pi[\alpha]}^{t,\mathbf{x},\alpha})\right],$$

$$\bar{V}(t, x, y, z) := \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} \mathbb{E}\left[F(X_\tau^{t,x,\alpha}, Y_\tau^{t,x,y,\alpha}, Z_\tau^{t,x,y,z,\alpha})\right] = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha})].$$

Given $\tau \in \mathcal{T}_{t,T}$, consider the function

$$(4.3.15) \qquad J(t, \mathbf{x}; \alpha, \tau) := \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha})].$$

Observing that $F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha}) = z + yF(\mathbf{X}_\tau^{t,x,1,0,\alpha})$, we have

$$(4.3.16) \qquad J(t, \mathbf{x}; \alpha, \tau) = z + yJ(t, (x, 1, 0); \alpha, \tau),$$

which in particular implies

$$(4.3.17) \qquad \bar{U}(t, x, y, z) = z + yU(t, x) \quad \bar{V}(t, x, y, z) = z + yV(t, x).$$

Thus, we can express the value functions U and V as

$$U(t, x) = \inf_{\pi \in \Pi_{t,T}} \sup_{\alpha \in \mathcal{A}_t} J(t, (x, 1, 0); \alpha, \pi[\alpha]), \quad V(t, x) = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} J(t, (x, 1, 0); \alpha, \tau).$$

The following result will be useful throughout this chapter.

Lemma 4.3.10. *Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\alpha \in \mathcal{A}$. For any $\theta \in \mathcal{T}_{t,T}$ and $\tau \in \mathcal{T}_{\theta,T}$,*

$$\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha}) \mid \mathcal{F}_\theta](\omega) = J\left(\theta(\omega), \mathbf{X}_\theta^{t,\mathbf{x},\alpha}(\omega); \alpha^{\theta,\omega}, \tau^{\theta,\omega}\right), \text{ for } \bar{\mathbb{P}}\text{-a.e. } \omega \in \Omega.$$

Proof. See Appendix C.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4.4 Supersolution Property of V

In this section, we will first study the following two functions

$$(4.4.1) \qquad G^\alpha(s, \mathbf{x}) := \inf_{\tau \in \mathcal{T}_{s,T}^s} J(s, \mathbf{x}; \alpha, \tau), \quad \widetilde{G}^\alpha(s, \mathbf{x}) := \inf_{\tau \in \mathcal{T}_{s,T}} J(s, \mathbf{x}; \alpha, \tau),$$

for $(s, \mathbf{x}) \in [0, T] \times \mathcal{S}$, where $\alpha \in \mathcal{A}$ is being fixed. A continuity result of G^α enables us to adapt the arguments in [24] to current context. We therefore obtain a weak dynamic programming principle (WDPP) for the function V (Proposition 4.4.2), which in turn leads to the supersolution property of V (Proposition 4.4.5).

Lemma 4.4.1. *Fix $\alpha \in \mathcal{A}$.*

(i) *$\widetilde{G}^\alpha$ is continuous on $[0, T] \times \mathcal{S}$.*

(ii) *Suppose $\alpha \in \mathcal{A}_t$ for some $t \in [0, T]$. Then $G^\alpha = \widetilde{G}^\alpha$ on $[0, t] \times \mathcal{S}$. As a result, G^α is continuous on $[0, t] \times \mathcal{S}$.*

Proof. **(i)** For any $s \in [0, T]$ and $\mathbf{x} = (x, y, z) \in \mathcal{S}$, observe from (4.3.16) that $\widetilde{G}^\alpha(s, \mathbf{x}) = z + y\widetilde{G}^\alpha(s, (x, 1, 0))$. Thus, it is enough to prove that $\widetilde{G}^\alpha(s, (x, 1, 0))$ is continuous on $[0, T] \times \mathbb{R}^d$. Also note that under (4.2.4), we have

$$\widetilde{G}^\alpha(s, \mathbf{x}) = \inf_{\tau \in \mathcal{T}_{s,T}} J(s, \mathbf{x}; \alpha, \tau) = \inf_{\tau \in \mathcal{T}_{0,T}} J(s, \mathbf{x}; \alpha, \tau).$$

Now, for any $(s, x) \in [0, T] \times \mathbb{R}^d$, take an arbitrary sequence $\{(s_n, x_n)\}_{n \in \mathbb{N}} \subset [0, T] \times \mathbb{R}^d$ such that $(s_n, x_n) \to (s, x)$. Then the continuity of $\widetilde{G}^\alpha(s, (x, 1, 0))$ can be seen from the following estimation

$$\left| \widetilde{G}^\alpha(s_n, (x_n, 1, 0)) - \widetilde{G}^\alpha(s, (x, 1, 0)) \right|$$

$$= \left| \inf_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}[F(\mathbf{X}_\tau^{s_n, x_n, 1, 0, \alpha})] - \inf_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}[F(\mathbf{X}_\tau^{s, x, 1, 0, \alpha})] \right|$$

$$\leq \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left[\left| F(\mathbf{X}_\tau^{s_n, x_n, 1, 0, \alpha}) - F(\mathbf{X}_\tau^{s, x, 1, 0, \alpha}) \right| \right]$$

$$\leq \mathbb{E}\left[\sup_{0 \leq r \leq T} \left| F(\mathbf{X}_r^{s_n, x_n, 1, 0, \alpha}) - F(\mathbf{X}_r^{s, x, 1, 0, \alpha}) \right| \right] \to 0,$$

where the convergence follows from Lemma 4.3.9.

(ii) Suppose $\alpha \in \mathcal{A}_t$ for some $t \in [0, T]$. For any $(s, \mathbf{x}) \in [0, t] \times \mathcal{S}$ and $\tau \in \mathcal{T}_{s,T}$, by taking $\theta = s$ in Lemma 4.3.10, we have

$$J(s, \mathbf{x}; \alpha, \tau) = \mathbb{E}\left[\mathbb{E}[F(\mathbf{X}_\tau^{s,\mathbf{x},\alpha}) \mid \mathcal{F}_s](\omega)\right] = \mathbb{E}\left[J(s, \mathbf{x}; \alpha, \tau^{s,\omega})\right]$$

(4.4.2)

$$\geq \inf_{\tau \in \mathcal{T}_{s,T}^s} J(s, \mathbf{x}; \alpha, \tau),$$

where in the second equality we replace $\alpha^{s,\omega}$ by α, thanks to Proposition 4.2.9. We then conclude

(4.4.3) $$\inf_{\tau \in \mathcal{T}_{s,T}} J(s, \mathbf{x}; \alpha, \tau) = \inf_{\tau \in \mathcal{T}_{s,T}^s} J(s, \mathbf{x}; \alpha, \tau),$$

as the "$\leq$" relation is trivial. That is, $\widetilde{G}^\alpha(s, \mathbf{x}) = G^\alpha(s, \mathbf{x})$. $\qquad\square$

Now, we want to modify the arguments in the proof of [24, Theorem 3.5] to get a weak dynamic programming principle for V. Given $w : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}$, we mimic the relation between V and $\bar{V}$ in (4.3.17) and define $\bar{w} : [0, T] \times \mathcal{S} \mapsto \mathbb{R}$ by

(4.4.4) $$\bar{w}(t, x, y, z) := z + yw(t, x), \quad (t, x, y, z) \in [0, T] \times \mathcal{S}.$$

Proposition 4.4.2. *Fix* $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ *and* $\varepsilon > 0$. *Take arbitrary* $\alpha \in \mathcal{A}_t$, $\theta \in \mathcal{T}_{t,T}^t$ *and* $\varphi \in \mathrm{USC}([0, T] \times \mathbb{R}^d)$ *with* $\varphi \leq V$. *We have the following:*

(i) $\mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \infty$;

(ii) *If, moreover,* $\mathbb{E}[\bar{\varphi}^-(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \infty$, *then there exists* $\alpha^* \in \mathcal{A}_t$ *with* $\alpha_s^* = \alpha_s$ *for* $s \in [0, \theta)$ *such that*

$$\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^*})] \geq \mathbb{E}[Y_{\tau \wedge \theta}^{t,x,y,\alpha} \varphi(\tau \wedge \theta, X_{\tau \wedge \theta}^{t,x,\alpha}) + Z_{\tau \wedge \theta}^{t,x,y,z,\alpha}] - 4\varepsilon, \quad \forall \tau \in \mathcal{T}_{t,T}^t.$$

Proof. (i) First, observe that for any $\mathbf{x} = (x, y, z) \in \mathcal{S}$, $\bar{\varphi}(t, \mathbf{x}) = y\varphi(t, x) + z \leq yV(t, x) + z \leq yg(x) + z$, which implies $\bar{\varphi}^+(t, \mathbf{x}) \leq yg(x) + z$. It follows that

$$\begin{aligned}
\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) &\leq Y_\theta^{t,x,y,\alpha} g(X_\theta^{t,x,\alpha}) + Z_\theta^{t,x,y,z,\alpha} \\
&\leq Y_\theta^{t,x,y,\alpha} g(X_\theta^{t,x,\alpha}) + z + \int_t^\theta Y_s^{t,x,y,\alpha} f(s, X_s^{t,x,\alpha}, \alpha_s) ds,
\end{aligned}$$

the right-hand-side is integrable as a result of (4.3.9).

(ii) For each $(s, \eta) \in [0, T] \times \mathcal{S}$, by the definition of $\bar{V}$, there exists $\alpha^{(s,\eta),\varepsilon} \in \mathcal{A}_s$ such that

$$(4.4.5) \qquad \inf_{\tau \in \mathcal{T}_{s,T}^s} J(s, \eta; \alpha^{(s,\eta),\varepsilon}, \tau) \geq \bar{V}(s, \eta) - \varepsilon.$$

Note that $\varphi \in \mathrm{USC}([0, T] \times \mathbb{R}^d)$ implies $\bar{\varphi} \in \mathrm{USC}([0, T] \times \mathcal{S})$. Then by the upper semicontinuity of $\bar{\varphi}$ on $[0, T] \times \mathcal{S}$ and the lower semicontinuity of $G^{\alpha^{(s,\eta),\varepsilon}}$ on $[0, s] \times \mathcal{S}$ (from Lemma 4.4.1 (ii)), there must exist $r^{(s,\eta)} > 0$ such that

$$\bar{\varphi}(t', x') - \bar{\varphi}(s, \eta) \leq \varepsilon \text{ and } G^{\alpha^{(s,\eta),\varepsilon}}(s, \eta) - G^{\alpha^{(s,\eta),\varepsilon}}(t', x') \leq \varepsilon, \text{ for } (t', x') \in B(s, \eta; r^{(s,\eta)}),$$

where $B(s, \eta; r) = \{(t', x') \in [0, T] \times \mathcal{S} \mid t' \in (s - r, s], |x' - \eta| < r\}$, defined as in (4.1.2). It follows that if $(t', x') \in B(s, \eta; r^{(s,\eta)})$, we have

$$G^{\alpha^{(s,\eta),\varepsilon}}(t', x') \geq G^{\alpha^{(s,\eta),\varepsilon}}(s, \eta) - \varepsilon \geq \bar{V}(s, \eta) - 2\varepsilon \geq \bar{\varphi}(s, \eta) - 2\varepsilon \geq \bar{\varphi}(t', x') - 3\varepsilon,$$

where the second inequality is due to (4.4.5). Here, we do not use the usual topology induced by balls of the form $B_r(s, \eta) = \{(t', x') \in [0, T] \times \mathcal{S} \mid |t' - s| < r, |x' - \eta| < r\}$; instead, for the time variable, we consider the topology induced by half-closed intervals on $[0, T]$, i.e. the so-called upper limit topology (see e.g. [34, Ex.4 on p.66]). Note from [34, Ex.3 on p.174] and [86, Ex.3 on p.192] that $(0, T]$ is a Lindelöf space under this topology. It follows that, under this setting, $\{B(s, \eta; r) \mid (s, \eta) \in [0, T] \times \mathcal{S}, 0 < r \leq r^{(s,\eta)}\}$ forms an open covering of $(0, T] \times \mathcal{S}$, and there exists a countable subcovering $\{B(t_i, x_i; r_i)\}_{i \in \mathbb{N}}$ of $(0, T] \times \mathcal{S}$. Now set $A_0 := \{T\} \times \mathcal{S}$, $C_{-1} := \emptyset$ and define for all $i \in \mathbb{N} \cup \{0\}$

$$A_{i+1} := B(t_{i+1}, x_{i+1}; r_{i+1}) \setminus C_i, \text{ where } C_i := C_{i-1} \cup A_i.$$

Under this construction, we have

(4.4.6)
$$(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) \in \cup_{i \in \mathbb{N} \cup \{0\}} A_i \ \mathbb{P}\text{-a.s.}, \ A_i \cap A_j = \emptyset \text{ for } i \neq j,$$

$$G^{\alpha^{i,\varepsilon}}(t', x') \geq \bar{\varphi}(t', x') - 3\varepsilon \text{ for } (t', x') \in A_i, \text{ where } \alpha^{i,\varepsilon} := \alpha^{(t_i, x_i), \varepsilon}.$$

For any $n \in \mathbb{N}$, set $A^n := \cup_{0 \leq i \leq n} A_i$ and define

$$\alpha^{\varepsilon,n} := \alpha 1_{[0,\theta)} + \left(\alpha 1_{(A^n)^c}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) + \sum_{i=0}^n \alpha^{i,\varepsilon} 1_{A_i}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) \right) 1_{[\theta,T]} \in \mathcal{A}_t.$$

Note that $\alpha_s^{\varepsilon,n} = \alpha_s$ for $s \in [0, \theta)$. Whenever $\omega \in \{(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) \in A_i\}$, observe that

$(\alpha^{\varepsilon,n})^{\theta,\omega}(\omega') = \alpha^{\varepsilon,n} (\omega \otimes_\theta \phi_\theta(\omega')) = \alpha^{i,\varepsilon} (\omega \otimes_\theta \phi_\theta(\omega')) = (\alpha^{i,\varepsilon})^{\theta,\omega}(\omega')$; also, we have

$\alpha^{i,\varepsilon} \in \mathcal{A}_{\theta(\omega)}$, as $\alpha^{i,\varepsilon} \in \mathcal{A}_{t_i}$ and $\theta(\omega) \leq t_i$ on A_i. We then deduce from Lemma 4.3.10,

Proposition 4.2.9, and (4.4.6) that for $\bar{\bar{\mathbb{P}}}$-a.e. $\omega \in \Omega$

(4.4.7)
$$\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^{\varepsilon,n}}) 1_{\{\tau \geq \theta\}} | \mathcal{F}_\theta] 1_{A^n}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})$$

$$= 1_{\{\tau \geq \theta\}} \sum_{i=0}^n J(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}; \alpha^{i,\varepsilon}, \tau^{\theta,\omega}) 1_{A_i}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})$$

$$\geq 1_{\{\tau \geq \theta\}} \sum_{i=0}^n G^{\alpha^{i,\varepsilon}}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) 1_{A_i}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})$$

$$\geq 1_{\{\tau \geq \theta\}} [\bar{\varphi}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) - 3\varepsilon] 1_{A^n}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}).$$

Hence, we have

(4.4.8)

$$\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^{\varepsilon,n}})] = \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha}) 1_{\{\tau < \theta\}}] + \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^{\varepsilon,n}}) 1_{\{\tau \geq \theta\}}]$$

$$= \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha}) 1_{\{\tau < \theta\}}] + \mathbb{E}\left[\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^{\varepsilon,n}}) 1_{\{\tau \geq \theta\}} | \mathcal{F}_\theta] 1_{A^n}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) \right]$$

$$+ \mathbb{E}\left[\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^{\varepsilon,n}}) 1_{\{\tau \geq \theta\}} | \mathcal{F}_\theta] 1_{(A^n)^c}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) \right]$$

$$\geq \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha}) 1_{\{\tau < \theta\}}] + \mathbb{E}[1_{\{\tau \geq \theta\}} \bar{\varphi}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) 1_{A^n}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - 3\varepsilon$$

$$\geq \mathbb{E}[1_{\{\tau < \theta\}} \bar{\varphi}(\tau, \mathbf{X}_\tau^{t,\mathbf{x},\alpha})] + \mathbb{E}[1_{\{\tau \geq \theta\}} \bar{\varphi}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) 1_{A^n}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - 3\varepsilon,$$

where the first inequality comes from (4.4.7), and the second inequality is due to the

observation that

$$F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha}) = Y_\tau^{t,x,y,\alpha} g(X_\tau^{t,x,\alpha}) + Z_\tau^{t,x,y,z,\alpha} \geq Y_\tau^{t,x,y,\alpha} V(\tau, X_\tau^{t,x,\alpha}) + Z_\tau^{t,x,y,z,\alpha}$$

$$\geq Y_\tau^{t,x,y,\alpha} \varphi(\tau, X_\tau^{t,x,\alpha}) + Z_\tau^{t,x,y,z,\alpha}.$$

Since $\mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \infty$ (by part (i)), there exists $n^* \in \mathbb{N}$ such that

$$\mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - \mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) 1_{A^{n^*}}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \varepsilon.$$

We observe the following holds for any $\tau \in \mathcal{T}_{t,T}^t$

$$
\begin{aligned}
(4.4.9) \quad & \mathbb{E}[1_{\{\tau \geq \theta\}} \bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - \mathbb{E}[1_{\{\tau \geq \theta\}} \bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) 1_{A^{n^*}}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] \\
& \qquad \leq \mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - \mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) 1_{A^{n^*}}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \varepsilon.
\end{aligned}
$$

Suppose $\mathbb{E}[\bar{\varphi}^-(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \infty$, then we can conclude from (4.4.9) that for any $\tau \in \mathcal{T}_{t,T}^t$

$$\mathbb{E}[1_{\{\tau \geq \theta\}} \bar{\varphi}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] = \mathbb{E}[1_{\{\tau \geq \theta\}} \bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - \mathbb{E}[1_{\{\tau \geq \theta\}} \bar{\varphi}^-(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})]$$

$$\leq \mathbb{E}[1_{\{\tau \geq \theta\}} \bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) 1_{A^{n^*}}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] + \varepsilon - \mathbb{E}[1_{\{\tau \geq \theta\}} \bar{\varphi}^-(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) 1_{A^{n^*}}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})]$$

$$= \mathbb{E}[1_{\{\tau \geq \theta\}} \bar{\varphi}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) 1_{A^{n^*}}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] + \varepsilon.$$

Taking $\alpha^* = \alpha^{\varepsilon, n^*}$, we now conclude from (4.4.8) and the above inequality that

$$\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^*})] \geq \mathbb{E}[1_{\{\tau < \theta\}} \bar{\varphi}(\tau, \mathbf{X}_\tau^{t,\mathbf{x},\alpha})] + \mathbb{E}[1_{\{\tau \geq \theta\}} \bar{\varphi}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - 4\varepsilon$$

$$= \mathbb{E}[\bar{\varphi}(\tau \wedge \theta, \mathbf{X}_{\tau \wedge \theta}^{t,\mathbf{x},\alpha})] - 4\varepsilon$$

$$= E[Y_{\tau \wedge \theta}^{t,x,y,\alpha} \varphi(\tau \wedge \theta, X_{\tau \wedge \theta}^{t,x,\alpha}) + Z_{\tau \wedge \theta}^{t,x,y,z,\alpha}] - 4\varepsilon.$$

$\square$

We still need the following property of V to obtain the supersolution property.

Proposition 4.4.3. *For any* $(t, x) \in [0, T] \times \mathbb{R}^d$, $V(t, x) = \sup_{\alpha \in \mathcal{A}} \widetilde{G}^\alpha(t, (x, 1, 0))$.

Proof. Thanks to Lemma 4.4.1 (ii), we immediately have

$$V(t,x) = \sup_{\alpha \in \mathcal{A}_t} G^\alpha(t,(x,1,0)) = \sup_{\alpha \in \mathcal{A}_t} \widetilde{G}^\alpha(t,(x,1,0)) \leq \sup_{\alpha \in \mathcal{A}} \widetilde{G}^\alpha(t,(x,1,0)).$$

For the reverse inequality, fix $\alpha \in \mathcal{A}$ and $\mathbf{x} \in \mathcal{S}$. By a calculation similar to (4.4.2), we have $J(t,\mathbf{x};\alpha,\tau) = \mathbb{E}[J(t,\mathbf{x};\alpha^{t,\omega},\tau^{t,\omega})]$, for any $\tau \in \mathcal{T}_{t,T}$. Observing that $\tau^{t,\omega} \in \mathcal{T}_{t,T}^t$ for all $\tau \in \mathcal{T}_{t,T}$ (by Proposition 4.2.8), and that $\mathbb{E}[J(t,\mathbf{x};\alpha^{t,\omega},\tau^{t,\omega})] = \mathbb{E}[J(t,\mathbf{x};\alpha^{t,\omega},\tau)]$ for all $\tau \in \mathcal{T}_{t,T}^t$ (by Proposition 4.2.7), we obtain

$$
\begin{aligned}
\inf_{\tau \in \mathcal{T}_{t,T}} J(t,\mathbf{x};\alpha,\tau) &= \inf_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}[J(t,\mathbf{x};\alpha^{t,\omega},\tau^{t,\omega})] = \inf_{\tau \in \mathcal{T}_{t,T}^t} \mathbb{E}[J(t,\mathbf{x};\alpha^{t,\omega},\tau)] \\
&\leq \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} \mathbb{E}[J(t,\mathbf{x};\alpha,\tau)] = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} J(t,\mathbf{x};\alpha,\tau),
\end{aligned}
$$

where the inequality is due to the fact that $\alpha^{t,\omega} \in \mathcal{A}_t$. By setting $\mathbf{x} := (x,1,0)$ and taking supremum over $\alpha \in \mathcal{A}$, we get $\sup_{\alpha \in \mathcal{A}} \widetilde{G}^\alpha(t,(x,1,0)) \leq V(t,x)$. $\square$

Corollary 4.4.4. $V \in \mathrm{LSC}([0,T] \times \mathbb{R}^d)$.

Proof. By Proposition 4.4.3 and Lemma 4.4.1 (i), V is a supremum of a collection of continuous functions defined on $[0,T] \times \mathbb{R}^d$, and thus has to be lower semicontinuous on the same space. $\square$

Now, we are ready to present the main result of this section. Recall that the operator $\overline{H}$ is defined in (4.3.14).

Proposition 4.4.5. *The function V is a lower semicontinuous viscosity supersolution to the obstacle problem of a Hamilton-Jacobi-Bellman equation*

$$(4.4.10) \quad \max\left\{ c(t,x)w - \frac{\partial w}{\partial t} + \overline{H}(t,x,D_x w, D_x^2 w), \; w - g(x) \right\} = 0 \; \text{on} \; [0,T) \times \mathbb{R}^d,$$

and satisfies the polynomial growth condition: there exists $N > 0$ such that

$$(4.4.11) \qquad |V(t,x)| \leq N(1+|x|^{\bar{p}}), \; \forall (t,x) \in [0,T] \times \mathbb{R}^d.$$

Proof. The lower semicontinuity of V was shown in Corollary 4.4.4. Observe that

$$0 \le V(t, x) \le \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}[F(\mathbf{X}_T^{t,x,1,0,\alpha})] \le \sup_{\alpha \in \mathcal{A}} \mathbb{E}[F(\mathbf{X}_T^{t,x,1,0,\alpha})] =: v(t, x). \text{ Since } v$$

satisfies (4.4.11) as a result of [77, Theorem 3.1.5], so does V.

To prove the supersolution property, let $h \in C^{1,2}([0, T) \times \mathbb{R}^d)$ be such that

$$(4.4.12) \quad 0 = (V - h)(t_0, x_0) < (V - h)(t, x), \ \forall \ (t, x) \in ([0, T) \times \mathbb{R}^d) \setminus \{(t_0, x_0)\},$$

for some $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$. If $V(t_0, x_0) = g(x_0)$, then there is nothing to prove. We, therefore, assume that $V(t_0, x_0) < g(x_0)$. For such (t_0, x_0) it is enough to prove the following inequality:

$$0 \le c(t_0, x_0)h(t_0, x_0) - \frac{\partial h}{\partial t}(t_0, x_0) + \overline{H}(\cdot, D_x h, D_x^2 h)(t_0, x_0).$$

Assume the contrary. Then, by the definition of $\overline{H}$ in (4.3.14), there must exist $\zeta_0 \in M$ such that

$$0 > c(t_0, x_0)h(t_0, x_0) - \frac{\partial h}{\partial t}(t_0, x_0) + (H^{\zeta_0})^*(\cdot, D_x h, D_x^2 h)(t_0, x_0).$$

Moreover, from the upper semicontinuity of $(H^{\zeta_0})^*$ and the fact that $(H^{\zeta_0})^* \ge H^{\zeta_0}$, we can choose some $r > 0$ with $t_0 + r < T$ such that

$$(4.4.13) \quad 0 > c(t, x)h(t, x) - \frac{\partial h}{\partial t}(t, x) + H^{\zeta_0}(\cdot, D_x h, D_x^2 h)(t, x), \ \forall \ (t, x) \in \bar{B}_r(t_0, x_0).$$

Define $\zeta \in \mathcal{A}$ by setting $\zeta_t = \zeta_0$ for all $t \ge 0$, and introduce the stopping time

$$\theta := \inf \left\{ s \ge t_0 \ | \ (s, X_s^{t_0, x_0, \zeta}) \notin B_r(t_0, x_0) \right\} \in \mathcal{T}_{t_0, T}^{t_0}.$$

Note that we have $\theta \in \mathcal{T}_{t_0, T}^{t_0}$ as the control ζ is by definition independent of $\mathcal{F}_{t_0}$. Now, by applying the product rule of stochastic calculus to $Y_s^{t_0, x_0, 1, \zeta} h(s, X_s^{t_0, x_0, \zeta})$ and

recalling (4.4.13) and $c \leq \bar{c}$, we obtain that for any $\tau \in \mathcal{T}_{t_0,T}^{t_0}$,

$$V(t_0, x_0) = h(t_0, x_0)$$

$$= \mathbb{E}\left[Y_{\theta \wedge \tau}^{t_0,x_0,1,\zeta} h(\theta \wedge \tau, X_{\theta \wedge \tau}^{t_0,x_0,\zeta}) \right.$$

$$\left. + \int_{t_0}^{\theta \wedge \tau} Y_s^{t_0,x_0,1,\zeta} \left(ch - \frac{\partial h}{\partial t} + H^{\zeta_0}(\cdot, D_x h, D_x^2 h) + f \right)(s, X_s^{t_0,x_0,\zeta}, \zeta_0) ds \right]$$

$$(4.4.14) \quad < \mathbb{E}\left[Y_{\theta \wedge \tau}^{t_0,x_0,1,\zeta} h(\theta \wedge \tau, X_{\theta \wedge \tau}^{t_0,x_0,\zeta}) + \int_{t_0}^{\theta \wedge \tau} Y_s^{t_0,x_0,1,\zeta} f(s, X_s^{t_0,x_0,\zeta}, \zeta_0) ds \right].$$

In the following, we will work towards a contradiction to (4.4.14). First, define

$$\bar{h}(\theta, \mathbf{X}_\theta^{t_0,x_0,1,0,\zeta}) := Y_\theta^{t_0,x_0,1,\zeta} h(\theta, X_\theta^{t_0,x_0,\zeta}) + \int_{t_0}^\theta Y_s^{t_0,x_0,1,\zeta} f(s, X_s^{t_0,x_0,\zeta}, \zeta_0) ds.$$

Note from (4.4.14) that $\mathbb{E}[\bar{h}(\theta, \mathbf{X}_\theta^{t_0,x_0,1,0,\zeta})]$ is bounded from below. It follows from

this fact that $\mathbb{E}[\bar{h}^-(\theta, \mathbf{X}_\theta^{t_0,x_0,1,0,\zeta})] < \infty$, as we already have $\mathbb{E}[\bar{h}^+(\theta, \mathbf{X}_\theta^{t_0,x_0,1,0,\zeta})] < \infty$

from Proposition 4.4.2 (i). For each $n \in \mathbb{N}$, we can therefore apply Proposition 4.4.2

(ii) and conclude that there exists $\alpha^{*,n} \in \mathcal{A}_{t_0}$, with $\alpha_s^{*,n} = \zeta_s$ for all $s \leq \theta$, such that

for any $\tau \in \mathcal{T}_{t_0,T}^{t_0}$,

$$(4.4.15) \quad \begin{aligned} &\mathbb{E}[F(\mathbf{X}_\tau^{t_0,x_0,1,0,\alpha^{*,n}})] \\ &\geq \mathbb{E}\left[Y_{\theta \wedge \tau}^{t_0,x_0,1,\zeta} h(\theta \wedge \tau, X_{\theta \wedge \tau}^{t_0,x_0,\zeta}) + \int_{t_0}^{\theta \wedge \tau} Y_s^{t_0,x_0,1,\zeta} f(s, X_s^{t_0,x_0,\zeta}, \zeta_0) ds \right] - \frac{1}{n}. \end{aligned}$$

Next, thanks to the definition of V and the classical theory of Snell envelopes (see

e.g. Appendix D, and especially Theorem D.12, in [67]), we have

$$(4.4.16) \qquad V(t_0, x_0) \geq G^{\alpha^{*,n}}(t_0, (x_0, 1, 0)) = \mathbb{E}[F(\mathbf{X}_{\tau^n}^{t_0,x_0,1,0,\alpha^{*,n}})],$$

where

$$\tau^n := \inf\left\{ s \geq t_0 \mid G^{\alpha^{*,n}}(s, \mathbf{X}_s^{t_0,x_0,1,0,\alpha^{*,n}}) = g(X_s^{t_0,x_0,\alpha^{*,n}}) \right\} \in \mathcal{T}_{t_0,T}^{t_0}.$$

Note that we may apply [67, Theorem D.12] because (4.3.9) holds. Combining

(4.4.16) and (4.4.15), we obtain

$$V(t_0, x_0) \geq \mathbb{E}\left[Y_{\theta \wedge \tau^n}^{t_0,x_0,1,\zeta} h(\theta \wedge \tau^n, X_{\theta \wedge \tau^n}^{t_0,x_0,\zeta}) + \int_{t_0}^{\theta \wedge \tau^n} Y_s^{t_0,x_0,1,\zeta} f(s, X_s^{t_0,x_0,\zeta}, \zeta_0) ds \right] - \frac{1}{n}.$$

By sending n to infinity and using Fatou's Lemma, we conclude that

$$V(t_0, x_0) \geq \mathbb{E}\left[Y_{\theta \wedge \tau^*}^{t_0, x_0, 1, \zeta} h(\theta \wedge \tau^*, X_{\theta \wedge \tau^*}^{t_0, x_0, \zeta}) + \int_{t_0}^{\theta \wedge \tau^*} Y_s^{t_0, x_0, 1, \zeta} f(s, X_s^{t_0, x_0, \zeta}, \zeta_0) ds\right],$$

where $\tau^* := \liminf_{n \to \infty} \tau^n$ is a stopping time in $\mathcal{T}_{t_0, T}^{t_0}$, thanks to the right continuity of the filtration $\mathbb{F}^{t_0}$. The above inequality, however, contradicts (4.4.14). $\qquad \square$

Remark 4.4.6. The lower semicontinuity of V is needed for the proof of Proposition 4.4.5. To see this, suppose V is not lower semicontinuous. Then V should be replaced by V_* in (4.4.12) and (4.4.14). The last inequality in the proof and (4.4.14) would then yield $V_*(t_0, x_0) < V(t_0, x_0)$, which is not a contradiction.

Remark 4.4.7. Due to the lack of continuity in t of the functions b, σ, and f, we use $\overline{H}$, instead of H^*, in (4.4.10). If we were using H^*, we in general would not be able to find a $\zeta_0 \in M$ such that (4.4.13) holds (due to the lack of continuity in t). If b, σ, and f are actually continuous in t, then we see from (4.3.13) and (4.3.14) that $\overline{H} = H = H^*$.

4.5 Subsolution Property of U^*

As in Section 4.4, we will first prove a continuity result (Lemma 4.5.5), which leads to a weak dynamic programming principle for U (Proposition 4.5.6). Then, we will show that the subsolution property of U^* follows from this weak dynamic programming principle (Proposition 4.5.7). Remember that U^* is the upper semicontinuous envelope of U defined as in (4.1.3).

Fix $s \in [0, T]$ and $\xi \in L_d^p(\Omega, \mathcal{F}_s)$ for some $p \in [1, \infty)$. For any $\alpha \in \mathcal{A}$ and $\pi_1, \pi_2 \in \Pi_{s,T}$ with $\pi_1[\beta] \leq \pi_2[\beta]$ $\overline{\mathbb{P}}$-a.s. for all $\beta \in \mathcal{A}$, we define

$$(4.5.1) \qquad \mathcal{B}_{\pi_1}^{s, \xi, \alpha} := \left\{\beta \in \mathcal{A} \;\middle|\; \int_s^{\pi_1[\alpha]} \rho'(\beta_u, \alpha_u) du = 0 \;\; \overline{\mathbb{P}}\text{-a.s.}\right\},$$

and introduce the random variable

(4.5.2)

$$K^{s,\xi,\alpha}(\pi_1, \pi_2) :=$$

$$\text{ess sup}_{\beta \in \mathcal{B}_{\pi_1}^{s,\xi,\alpha}} \mathbb{E} \left[\int_{\pi_1[\alpha]}^{\pi_2[\beta]} Y_u^{\pi_1[\alpha], X_{\pi_1[\alpha]}^{s,\xi,\beta}, 1, \beta} f(u, X_u^{s,\xi,\beta}, \beta_u) du + Y_{\pi_2[\beta]}^{\pi_1[\alpha], X_{\pi_1[\alpha]}^{s,\xi,\beta}, 1, \beta} g(X_{\pi_2[\beta]}^{s,\xi,\beta}) \,\middle|\, \mathcal{F}_{\pi_1[\alpha]} \right].$$

Observe from the definition of $\mathcal{B}_{\pi_1}^{s,\xi,\alpha}$ and Definition 4.3.1 (i) that

(4.5.3) $$\pi_1[\beta] = \pi_1[\alpha] \quad \overline{\mathbb{P}}\text{-a.s.} \quad \forall \beta \in \mathcal{B}_{\pi_1}^{s,\xi,\alpha}.$$

This in particular implies $\pi_2[\beta] \geq \pi_1[\beta] = \pi_1[\alpha]$ $\overline{\mathbb{P}}$-a.s. $\forall \beta \in \mathcal{B}_{\pi_1}^{s,\xi,\alpha}$, which shows that $K^{s,\xi,\alpha}(\pi_1, \pi_2)$ is well-defined. Given any constant strategies $\pi_1[\cdot] \equiv \tau_1 \in \mathcal{T}_{s,T}^s$ and $\pi_2[\cdot] \equiv \tau_2 \in \mathcal{T}_{s,T}^s$, we will simply write $K^{s,\xi,\alpha}(\pi_1, \pi_2)$ as $K^{s,\xi,\alpha}(\tau_1, \tau_2)$. For the particular case where $\xi = x \in \mathbb{R}^d$, we also consider

$$\Gamma^{s,x,\alpha}(\pi_1, \pi_2) := \int_s^{\pi_1[\alpha]} Y_u^{s,x,1,\alpha} f(u, X_u^{s,x,\alpha}, \alpha_u) du + Y_{\pi_1[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_1, \pi_2).$$

Remark 4.5.1. Let us write $K^{s,x,\alpha}(\pi_1, \pi_2) = \text{ess sup}_{\beta \in \mathcal{B}_{\pi_1}^{s,x,\alpha}} \mathbb{E}[R_{\pi_1,\pi_2}^{s,x,\alpha}(\beta) \mid \mathcal{F}_{\pi_1[\alpha]}]$ for simplicity. Note that the set of random variables $\{\mathbb{E}[R_{\pi_1,\pi_2}^{s,x,\alpha}(\beta) \mid \mathcal{F}_{\pi_1[\alpha]}]\}_{\beta \in \mathcal{B}_{\pi_1}^{s,x,\alpha}}$ is closed under pairwise maximization. Indeed, given $\beta_1, \beta_2 \in \mathcal{B}_{\pi_1}^{s,x,\alpha}$, set $A := \{\mathbb{E}[R_{\pi_1,\pi_2}^{s,x,\alpha}(\beta_1) \mid \mathcal{F}_{\pi_1[\alpha]}] \geq \mathbb{E}[R_{\pi_1,\pi_2}^{s,x,\alpha}(\beta_2) \mid \mathcal{F}_{\pi_1[\alpha]}]\} \in \mathcal{F}_{\pi_1[\alpha]}$ and define $\beta_3 := \beta_1 1_{[0,\pi_1[\alpha])} + (\beta_1 1_A + \beta_2 1_{A^c}) 1_{[\pi_1[\alpha],T]} \in \mathcal{B}_{\pi_1}^{s,x,\alpha}$. Then, observe that

$$\mathbb{E}[R_{\pi_1,\pi_2}^{s,x,\alpha}(\beta_3) \mid \mathcal{F}_{\pi_1[\alpha]}] = \mathbb{E}[R_{\pi_1,\pi_2}^{s,x,\alpha}(\beta_1) \mid \mathcal{F}_{\pi_1[\alpha]}] 1_A + \mathbb{E}[R_{\pi_1,\pi_2}^{s,x,\alpha}(\beta_2) \mid \mathcal{F}_{\pi_1[\alpha]}] 1_{A^c}$$

$$= \mathbb{E}[R_{\pi_1,\pi_2}^{s,x,\alpha}(\beta_1) \mid \mathcal{F}_{\pi_1[\alpha]}] \vee \mathbb{E}[R_{\pi_1,\pi_2}^{s,x,\alpha}(\beta_2) \mid \mathcal{F}_{\pi_1[\alpha]}].$$

Thus, we conclude from Theorem A.3 in [67, Appendix A] that there exists a sequence $\{\beta^n\}_{n \in \mathbb{N}}$ in $\mathcal{B}_{\pi_1}^{s,x,\alpha}$ such that $K^{s,x,\alpha}(\pi_1, \pi_2) = \uparrow \lim_{n \to \infty} \mathbb{E}[R_{\pi_1,\pi_2}^{s,x,\alpha}(\beta^n) \mid \mathcal{F}_{\pi_1[\alpha]}]$ $\overline{\mathbb{P}}$-a.s.

Lemma 4.5.2. *Fix $(s, x) \in [0, T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}$. For any $r \in [s, T]$ and $\pi \in \Pi_{r,T}$,*

$$K^{s,x,\alpha}(r, \pi) = K^{r, X_r^{s,x,\alpha}, \alpha}(r, \pi) \quad \overline{\mathbb{P}}\text{-a.s.}$$

Proof. For any $\beta \in \mathcal{B}_r^{s,x,\alpha}$, we see from Remark 4.2.6 (i) that $X_u^{s,x,\beta} = X_u^{r,X_r^{s,x,\alpha},\beta}$ for $u \in [r,T]$ $\overline{\mathbb{P}}$-a.s. It follows from (4.5.2) that

$$K^{s,x,\alpha}(r,\pi) =$$

$$\text{ess sup}_{\beta \in \mathcal{B}_r^{s,x,\alpha}} \mathbb{E}\left[\int_r^{\pi[\beta]} Y_u^{r,X_r^{s,x,\alpha},1,\beta} f(u, X_u^{r,X_r^{s,x,\alpha},\beta}, \beta_u) du + Y_{\pi[\beta]}^{r,X_r^{s,x,\alpha},1,\beta} g(X_{\pi[\beta]}^{r,X_r^{s,x,\alpha},\beta}) \,\bigg|\, \mathcal{F}_r\right].$$

Observing from (4.5.1) that $\mathcal{B}_r^{s,x,\alpha} \subseteq \mathcal{A} = \mathcal{B}_r^{r,X_r^{s,x,\alpha},\alpha}$, we conclude $K^{s,x,\alpha}(r,\pi) \leq K^{r,X_r^{s,x,\alpha},\alpha}(r,\pi)$. On the other hand, for any $\beta \in \mathcal{A}$, define $\bar{\beta} := \alpha 1_{[0,r)} + \beta 1_{[r,T]} \in \mathcal{B}_r^{s,x,\alpha}$. Then, by Remark 4.2.6 (i) again, we have $X_u^{s,x,\bar{\beta}} = X_u^{r,X_r^{s,x,\alpha},\beta}$ for $u \in [r,T]$ $\overline{\mathbb{P}}$-a.s. Also, we have $\pi[\bar{\beta}] = \pi[\beta]$, thanks to Definition 4.3.1 (i). Therefore,

$$\mathbb{E}\left[\int_r^{\pi[\beta]} Y_u^{r,X_r^{s,x,\alpha},1,\beta} f(u, X_u^{r,X_r^{s,x,\alpha},\beta}, \beta_u) du + Y_{\pi[\beta]}^{r,X_r^{s,x,\alpha},1,\beta} g(X_{\pi[\beta]}^{r,X_r^{s,x,\alpha},\beta}) \,\bigg|\, \mathcal{F}_r\right]$$

$$= \mathbb{E}\left[\int_r^{\pi[\bar{\beta}]} Y_u^{r,X_r^{s,x,\bar{\beta}},1,\bar{\beta}} f(u, X_u^{s,x,\bar{\beta}}, \bar{\beta}_u) du + Y_{\pi[\bar{\beta}]}^{r,X_r^{s,x,\bar{\beta}},1,\bar{\beta}} g(X_{\pi[\bar{\beta}]}^{s,x,\bar{\beta}}) \,\bigg|\, \mathcal{F}_r\right].$$

In view of (4.5.2), this implies $K^{r,X_r^{s,x,\alpha},\alpha}(r,\pi) \leq K^{s,x,\alpha}(r,\pi)$. $\qquad\square$

Lemma 4.5.3. *Fix $(s,x) \in [0,T] \times \mathbb{R}^d$. Given $\alpha \in \mathcal{A}$ and $\pi_1, \pi_2, \pi_3 \in \Pi_{s,T}$ with $\pi_1[\beta] \leq \pi_2[\beta] \leq \pi_3[\beta]$ $\overline{\mathbb{P}}$-a.s. for all $\beta \in \mathcal{A}$, it holds $\overline{\mathbb{P}}$-a.s. that*

$$\mathbb{E}\left[\int_{\pi_1[\alpha]}^{\pi_2[\alpha]} Y_u^{s,x,1,\alpha} f(u, X_u^{s,x,\alpha}, \alpha_u) du + Y_{\pi_2[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_2, \pi_3) \,\bigg|\, \mathcal{F}_{\pi_1[\alpha]}\right]$$

$$\leq Y_{\pi_1[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_1, \pi_3).$$

Moreover, we have the following supermartingale property:

$$\mathbb{E}[\Gamma^{s,x,\alpha}(\pi_2, \pi_3) \mid \mathcal{F}_{\pi_1[\alpha]}] \leq \Gamma^{s,x,\alpha}(\pi_1, \pi_3) \ \overline{\mathbb{P}}\text{-a.s.}$$

Proof. By Remark 4.5.1, there exists a sequence $\{\beta^n\}_{n \in \mathbb{N}}$ in $\mathcal{B}_{\pi_2}^{s,x,\alpha}$ such that

$$K^{s,x,\alpha}(\pi_2, \pi_3) = \uparrow \lim_{n \to \infty} \mathbb{E}[R_{\pi_2,\pi_3}^{s,x,\alpha}(\beta^n) \mid \mathcal{F}_{\pi_2[\alpha]}] \ \overline{\mathbb{P}}\text{-a.s.}$$

From the definition of $\mathcal{B}_{\pi_2}^{s,x,\alpha}$ in (4.5.1), $\beta_u^n = \alpha_u$ for a.e. $u \in [s, \pi_2[\alpha])$ $\overline{\mathbb{P}}$-a.s. We can then compute as follows:

$$\mathbb{E}\left[Y_{\pi_2[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_2, \pi_3) \,\Big|\, \mathcal{F}_{\pi_1[\alpha]}\right]$$

$$= \mathbb{E}\left\{Y_{\pi_2[\alpha]}^{s,x,1,\alpha} \lim_{n\to\infty} \mathbb{E}\left[\int_{\pi_2[\alpha]}^{\pi_3[\beta^n]} Y_u^{\pi_2[\alpha],X_{\pi_2[\alpha]}^{s,x,\beta^n},1,\beta^n} f(u, X_u^{s,x,\beta^n}, \beta_u^n)du \right.\right.$$

$$\left.\left. + Y_{\pi_3[\beta^n]}^{\pi_2[\alpha],X_{\pi_2[\alpha]}^{s,x,\beta^n},1,\beta^n} g(X_{\pi_3[\beta^n]}^{s,x,\beta^n})\Big|\mathcal{F}_{\pi_2[\alpha]}\right]\Big|\mathcal{F}_{\pi_1[\alpha]}\right\}$$

$$= \mathbb{E}\left\{\lim_{n\to\infty} \mathbb{E}\left[\int_{\pi_2[\alpha]}^{\pi_3[\beta^n]} Y_u^{s,x,1,\beta^n} f(u, X_u^{s,x,\beta^n}, \beta_u^n)du + Y_{\pi_3[\beta^n]}^{s,x,1,\beta^n} g(X_{\pi_3[\beta^n]}^{s,x,\beta^n}) \,\Big|\, \mathcal{F}_{\pi_2[\alpha]}\right] \Big|\, \mathcal{F}_{\pi_1[\alpha]}\right\}$$

$$= \lim_{n\to\infty} \mathbb{E}\left[\int_{\pi_2[\alpha]}^{\pi_3[\beta^n]} Y_u^{s,x,1,\beta^n} f(u, X_u^{s,x,\beta^n}, \beta_u^n)du + Y_{\pi_3[\beta^n]}^{s,x,1,\beta^n} g(X_{\pi_3[\beta^n]}^{s,x,\beta^n}) \,\Big|\, \mathcal{F}_{\pi_1[\alpha]}\right],$$

where the last line follows from the monotone convergence theorem and the tower property for conditional expectations. We therefore conclude that

$$\mathbb{E}\left[\int_{\pi_1[\alpha]}^{\pi_2[\alpha]} Y_u^{s,x,1,\alpha} f(u, X_u^{s,x,\alpha}, \alpha_u)du + Y_{\pi_2[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_2, \pi_3) \,\Big|\, \mathcal{F}_{\pi_1[\alpha]}\right]$$

$$= \lim_{n\to\infty} \mathbb{E}\left[\int_{\pi_1[\alpha]}^{\pi_3[\beta^n]} Y_u^{s,x,1,\beta^n} f(u, X_u^{s,x,\beta^n}, \beta_u^n)du + Y_{\pi_3[\beta^n]}^{s,x,1,\beta^n} g(X_{\pi_3[\beta^n]}^{s,x,\beta^n}) \,\Big|\, \mathcal{F}_{\pi_1[\alpha]}\right]$$

$$= Y_{\pi_1[\alpha]}^{s,x,1,\alpha} \lim_{n\to\infty} \mathbb{E}\left[\int_{\pi_1[\alpha]}^{\pi_3[\beta^n]} Y_u^{\pi_1[\alpha],X_{\pi_1[\alpha]}^{s,x,\beta^n},1,\beta^n} f(u, X_u^{s,x,\beta^n}, \beta_u^n)du \right.$$

$$\left. + Y_{\pi_3[\beta^n]}^{\pi_1[\alpha],X_{\pi_1[\alpha]}^{s,x,\beta^n},1,\beta^n} g(X_{\pi_3[\beta^n]}^{s,x,\beta^n})\Big|\mathcal{F}_{\pi_1[\alpha]}\right]$$

$$\leq Y_{\pi_1[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_1, \pi_3),$$

where the inequality follows from the fact that $\beta^n \in \mathcal{B}_{\pi_2}^{s,x,\alpha} \subseteq \mathcal{B}_{\pi_1}^{s,x,\alpha}$. It follows that

$$\mathbb{E}[\Gamma^{s,x,\alpha}(\pi_2, \pi_3) \mid \mathcal{F}_{\pi_1[\alpha]}]$$

$$= \int_s^{\pi_1[\alpha]} Y_u^{s,x,1,\alpha} f(u, X_u^{s,x,\alpha}, \alpha_u)du$$

$$+ \mathbb{E}\left[\int_{\pi_1[\alpha]}^{\pi_2[\alpha]} Y_u^{s,x,1,\alpha} f(u, X_u^{s,x,\alpha}, \alpha_u)du + Y_{\pi_2[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_2, \pi_3) \,\Big|\, \mathcal{F}_{\pi_1[\alpha]}\right]$$

$$\leq \int_s^{\pi_1[\alpha]} Y_u^{s,x,1,\alpha} f(u, X_u^{s,x,\alpha}, \alpha_u)du + Y_{\pi_1[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_1, \pi_3) = \Gamma^{s,x,\alpha}(\pi_1, \pi_3).$$

$\square$

Lemma 4.5.4. *For any* $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ *and* $\pi \in \Pi_{t,T}$,

$$\sup_{\alpha \in \mathcal{A}} J(t, \mathbf{x}; \alpha, \pi[\alpha]) = \sup_{\alpha \in \mathcal{A}_t} J(t, \mathbf{x}; \alpha, \pi[\alpha]).$$

Proof. Fix $\alpha \in \mathcal{A}$ and $\mathbf{x} \in \mathcal{S}$. For any $\pi \in \Pi_{t,T}$, by taking $\theta = t$ in Lemma 4.3.10,

$$J(t, \mathbf{x}; \alpha, \pi[\alpha]) = \mathbb{E}\left[\mathbb{E}[F(\mathbf{X}_{\pi[\alpha]}^{t,\mathbf{x},\alpha}) \mid \mathcal{F}_t](\omega)\right]$$

$$= \mathbb{E}\left[J(t, \mathbf{x}; \alpha^{t,\omega}, \pi[\alpha^{t,\omega}])\right] \leq \sup_{\alpha \in \mathcal{A}_t} J(t, \mathbf{x}; \alpha, \pi[\alpha]).$$

Note that in the second equality we replace $\pi[\alpha]^{t,\omega}$ by $\pi[\alpha^{t,\omega}]$, thanks to Definition 4.3.1 (iii). Then, the last inequality holds as $\alpha^{t,\omega} \in \mathcal{A}_t$ for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$. Now, by taking supremum over $\alpha \in \mathcal{A}$, we have $\sup_{\alpha \in \mathcal{A}} J(t, \mathbf{x}; \alpha, \pi[\alpha]) \leq \sup_{\alpha \in \mathcal{A}_t} J(t, \mathbf{x}; \alpha, \pi[\alpha])$. Since the reverse inequality is trivial, this lemma follows. $\square$

Now, we are ready to state a continuity result for an optimal control problem.

Lemma 4.5.5. *Fix* $t \in [0, T]$. *For any* $\pi \in \Pi_{t,T}$, *the function* $L^\pi : [0, t] \times \mathcal{S}$ *defined by*

$$(4.5.4) \qquad\qquad L^\pi(s, \mathbf{x}) := \sup_{\alpha \in \mathcal{A}_s} J(s, \mathbf{x}; \alpha, \pi[\alpha])$$

is continuous.

Proof. Observing from (4.3.16) that $L^\pi(s, \mathbf{x}) = yL^\pi(s, (x, 1, 0)) + z$, it is enough to show the continuity of $L^\pi(s, (x, 1, 0))$ in (s, x) on $[0, t] \times \mathbb{R}^d$. By [77, Theorem 3.2.2], we know that $J(s, (x, 1, 0); \alpha, \tau)$ is continuous in x uniformly with respect to $s \in [0, t]$, $\alpha \in \mathcal{A}$, and $\tau \in \mathcal{T}_{t,T}$. This shows that the map $(s, x, \alpha) \mapsto J(s, (x, 1, 0); \alpha, \pi[\alpha])$ is continuous in x uniformly with respect to $s \in [0, t]$ and $\alpha \in \mathcal{A}$. Then, we see from the following estimation

$$\sup_{s \in [0,t]} |L^\pi(s, (x, 1, 0)) - L^\pi(s, (x', 1, 0))|$$

$$\leq \sup_{s \in [0,t]} \sup_{\alpha \in \mathcal{A}_s} |J(s, (x, 1, 0); \alpha, \pi[\alpha]) - J(s, (x', 1, 0); \alpha, \pi[\alpha])|$$

that $L^\pi(s, (x, 1, 0))$ is continuous in x uniformly with respect to $s \in [0, t]$. Thus, it suffices to prove that $L^\pi(s, (x, 1, 0))$ is continuous in s for each fixed x. To this end, we will first derive a dynamic programming principle for $L^\pi(s, (x, 1, 0))$, which corresponds to [77, Theorem 3.3.6]; the rest of the proof will then follow from the same argument in [77, Lemma 3.3.7].

Fix $(s, x) \in [0, t] \times \mathbb{R}^d$. Observe from (4.5.1) that $\mathcal{B}_s^{s,x,\alpha} = \mathcal{A}$ for all $\alpha \in \mathcal{A}$. In view of (4.5.2), this implies that $K^{s,x,\alpha}(s, \pi) = \text{ess sup}_{\beta \in \mathcal{A}} \mathbb{E}[F(\mathbf{X}_{\pi[\beta]}^{s,x,1,0,\beta}) \mid \mathcal{F}_s]$, which is independent of $\alpha \in \mathcal{A}$. We will therefore drop the superscript α in the rest of the proof. Now, we claim that $K^{s,x}(s, \pi)$ is deterministic and equal to $L^\pi(s, (x, 1, 0))$. First, since $\pi[\alpha] \in \mathcal{T}_{t,T}^s$ for all $\alpha \in \mathcal{A}_s$ (by Definition 4.3.1 (ii)), we observe from Lemma 4.3.10, Proposition 4.2.7 (ii), and Proposition 4.2.9 that

$$K^{s,x}(s, \pi) \geq \text{ess sup}_{\alpha \in \mathcal{A}_s} \mathbb{E}[F(\mathbf{X}_{\pi[\alpha]}^{s,x,1,0,\alpha}) \mid \mathcal{F}_s](\cdot)$$

(4.5.5)
$$= \text{ess sup}_{\alpha \in \mathcal{A}_s} J(s, (x, 1, 0); \alpha^{s,\cdot}, \pi[\alpha]^{s,\cdot})$$

$$= \sup_{\alpha \in \mathcal{A}_s} J(s, (x, 1, 0); \alpha, \pi[\alpha]) = L^\pi(s, (x, 1, 0)).$$

On the other hand, in view of Remark 4.5.1, there exists a sequence $\{\alpha^n\}_{n \in \mathbb{N}}$ in $\mathcal{A}$ such that $K^{s,x}(s, \pi) = \uparrow \lim_{n \to \infty} \mathbb{E}[F(\mathbf{X}_{\pi[\alpha^n]}^{s,x,1,0,\alpha^n}) \mid \mathcal{F}_s]$ $\mathbb{P}$-a.s. By the monotone convergence theorem,

$$\mathbb{E}[K^{s,x}(s, \pi)] = \mathbb{E}\left[\lim_{n \to \infty} \mathbb{E}[F(\mathbf{X}_{\pi[\alpha^n]}^{s,x,1,0,\alpha^n}) \mid \mathcal{F}_s]\right] = \lim_{n \to \infty} \mathbb{E}[F(\mathbf{X}_{\pi[\alpha^n]}^{s,x,1,0,\alpha^n})]$$

(4.5.6)
$$\leq \sup_{\alpha \in \mathcal{A}} \mathbb{E}[F(\mathbf{X}_{\pi[\alpha]}^{s,x,1,0,\alpha})] = L^\pi(s, (x, 1, 0)),$$

where the last equality is due to Lemma 4.5.4. From (4.5.5) and (4.5.6), we get $K^{s,x}(s, \pi) = L^\pi(s, (x, 1, 0))$. Then, for any $\alpha \in \mathcal{A}$, thanks to the supermartingale property introduced in Lemma 4.5.3, we have for all $r \in [s, t]$ that

$$L^\pi(s, (x, 1, 0)) = K^{s,x}(s, \pi) = \Gamma^{s,x,\alpha}(s, \pi) \geq \mathbb{E}[\Gamma^{s,x,\alpha}(r, \pi)]$$

$$\geq \mathbb{E}[\Gamma^{s,x,\alpha}(\pi, \pi)] \geq \mathbb{E}[F(\mathbf{X}_{\pi[\alpha]}^{s,x,1,0,\alpha})],$$

where the last equality follows from the fact that

$$K^{s,x,\alpha}(\pi,\pi) = \operatorname{ess\,sup}_{\beta\in\mathcal{B}_\pi^{s,x,\alpha}} g(X_{\pi[\beta]}^{s,x,\beta}) \geq g(X_{\pi[\alpha]}^{s,x,\alpha}) \;\;\overline{\mathbb{P}}\text{-a.s.};$$

see (4.5.2). By taking supremum over $\alpha \in \mathcal{A}$ and using Lemma 4.5.4, we obtain the following dynamic programming principle for $L^\tau(s,(x,1,0))$: for all $r \in [s,t]$,

$$
\begin{aligned}
L^\pi(s,(x,1,0)) &= \sup_{\alpha\in\mathcal{A}} \mathbb{E}[\Gamma^{s,x,\alpha}(r,\pi)] \\
&= \sup_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_s^r Y_u^{s,x,1,\alpha} f(u, X_u^{s,x,\alpha}, \alpha_u)du + Y_r^{s,x,1,\alpha} L^\pi(r,(X_r^{s,x,\alpha},1,0))\right],
\end{aligned}
$$

where the second equality follows from the fact $K^{s,x,\alpha}(r,\pi) = K^{r,X_r^{s,x,\alpha},\alpha}(r,\pi) = L^\pi(r,(X_r^{s,x,\alpha},1,0))$ $\overline{\mathbb{P}}$-a.s., as a consequence of Lemma 4.5.2. Now, we may apply the same argument in [77, Lemma 3.3.7] to show that $L^\pi(s,(x,1,0))$ is continuous in s on $[0,t]$. $\qquad\square$

Proposition 4.5.6. *Fix* $(t,\mathbf{x}) \in [0,T] \times \mathcal{S}$ *and* $\varepsilon > 0$. *For any* $\pi \in \Pi_{t,T}$ *and* $\varphi \in \mathrm{LSC}([0,T] \times \mathbb{R}^d)$ *with* $\varphi \geq U$, *there exists* $\pi^* \in \Pi_{t,T}$ *such that*

$$\mathbb{E}\left[F(\mathbf{X}_{\pi^*[\alpha]}^{t,\mathbf{x},\alpha})\right] \leq \mathbb{E}\left[Y_{\pi[\alpha]}^{t,x,y,\alpha}\varphi\left(\pi[\alpha], X_{\pi[\alpha]}^{t,x,\alpha}\right) + Z_{\pi[\alpha]}^{t,x,y,z,\alpha}\right] + 3\varepsilon, \quad \forall \alpha \in \mathcal{A}.$$

Proof. For each $(s,\eta) \in [0,T] \times \mathcal{S}$, by the definition of $\bar{U}$, there exists $\pi^{(s,\eta),\varepsilon} \in \Pi_{s,T}$ such that

$$(4.5.7) \qquad\qquad \sup_{\alpha\in\mathcal{A}_s} J\left(s,\eta;\alpha,\pi^{(s,\eta),\varepsilon}[\alpha]\right) \leq \bar{U}(s,\eta) + \varepsilon.$$

Recall the definition of $\bar\varphi$ in (4.4.4) and note that $\varphi \in \mathrm{LSC}([0,T] \times \mathbb{R}^d)$ implies $\bar\varphi \in \mathrm{LSC}([0,T] \times \mathcal{S})$. Then, by the lower semicontinuity of $\bar\varphi$ on $[0,T] \times \mathcal{S}$ and the upper semicontinuity of $L^{\pi^{(s,\eta),\varepsilon}}$ on $[0,s] \times \mathcal{S}$ (from Lemma 4.5.5), there must exist $r^{(s,\eta)} > 0$ such that

$$\bar\varphi(t',x') - \bar\varphi(s,\eta) \geq -\varepsilon \text{ and } L^{\pi^{(s,\eta),\varepsilon}}(t',x') - L^{\pi^{(s,\eta),\varepsilon}}(s,\eta) \leq \varepsilon,$$

for any (t', x') contained in the ball $B(s, \eta; r^{(s,\eta)})$, defined as in (4.1.2). It follows that if $(t', x') \in B(s, \eta; r^{(s,\eta)})$, we have

$$L^{\pi^{(s,\eta),\varepsilon}}(t', x') \leq L^{\pi^{(s,\eta),\varepsilon}}(s, \eta) + \varepsilon \leq \bar{U}(s, \eta) + 2\varepsilon \leq \bar{\varphi}(s, \eta) + 2\varepsilon \leq \bar{\varphi}(t', x') + 3\varepsilon,$$

where the second inequality is due to (4.5.7). By the same construction in the proof of Proposition 4.4.2, there exists a countable covering $\{B(t_i, x_i; r_i)\}_{i \in \mathbb{N}}$ of $(0, T] \times \mathcal{S}$, from which we can take a countable disjoint covering $\{A_i\}_{i \in \mathbb{N} \cup \{0\}}$ of $(0, T] \times \mathcal{S}$ such that

(4.5.8)
$$(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}) \in \cup_{i=1}^{\ell} A_i \;\; \overline{\mathbb{P}}\text{-a.s. } \forall \alpha \in \mathcal{A},$$

$$L^{\pi^{i,\varepsilon}}(t', x') \leq \bar{\varphi}(t', x') + 3\varepsilon \text{ for } (t', x') \in A_i, \text{ where } \pi^{i,\varepsilon} := \pi^{(t_i,x_i),\varepsilon}.$$

Now, define $\pi^* \in \Pi_{t,T}$ by

$$\pi^*[\alpha] := \sum_{i \geq 1} \pi^{i,\varepsilon}[\alpha] 1_{A_i}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}), \;\; \forall \alpha \in \mathcal{A}.$$

Fix $\alpha \in \mathcal{A}_t$. Observe that for $\overline{\mathbb{P}}$-a.e. $\omega \in \left\{ (\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}) \in A_i \right\} \subseteq \{\pi[\alpha] \leq t_i\}$, Definition 4.3.1 (iii) gives

(4.5.9)
$$(\pi^{i,\varepsilon}[\alpha])^{\pi[\alpha],\omega}(\omega') = \pi^{i,\varepsilon}[\alpha^{\pi[\alpha],\omega}](\omega') \text{ for } \overline{\mathbb{P}}\text{-a.e. } \omega' \in \Omega.$$

We then deduce from Lemma 4.3.10, (4.5.9), (4.5.4), and (4.5.8) that for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$,

$$\mathbb{E}\left[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\pi^*[\alpha]}) \;\middle|\; \mathcal{F}_{\pi[\alpha]} \right](\omega) \, 1_{A_i}(\pi[\alpha](\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}(\omega))$$

$$= J\left(\pi[\alpha](\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}(\omega); \alpha^{\pi[\alpha],\omega}, \pi^{i,\varepsilon}[\alpha^{\pi[\alpha],\omega}] \right) 1_{A_i}(\pi[\alpha](\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}(\omega))$$

$$\leq L^{\pi^{i,\varepsilon}}\left(\pi[\alpha](\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}(\omega) \right) 1_{A_i}(\pi[\alpha](\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}(\omega))$$

$$\leq \left[\bar{\varphi}\left(\pi[\alpha](\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}(\omega) \right) + 3\varepsilon \right] 1_{A_i}(\pi[\alpha](\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}(\omega)).$$

It follows from the monotone convergence theorem that

$$\mathbb{E}\left[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\pi^*[\alpha]}) \right] = \sum_{i \geq 1} \mathbb{E}\left[\mathbb{E}\left[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\pi^*[\alpha]}) \;\middle|\; \mathcal{F}_{\pi[\alpha]} \right] 1_{A_i}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}) \right]$$

$$\leq \mathbb{E}\left[\bar{\varphi}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}) \right] + 3\varepsilon,$$

which is the desired result by recalling again the definition of $\bar{\varphi}$ in (4.4.4). $\qquad\square$

The following is the main result of this section. Recall that the operator H is defined in (4.3.13), and H_* denotes the lower semicontinuous envelope of H defined as in (4.1.3).

Proposition 4.5.7. *The function U^* is a viscosity subsolution to the obstacle problem of a Hamilton-Jacobi-Bellman equation*

$$\max\left\{c(t,x)w - \frac{\partial w}{\partial t} + H_*(t, x, D_x w, D_x^2 w), w - g(x)\right\} = 0 \ on \ [0, T) \times \mathbb{R}^d,$$

and satisfies the polynomial growth condition (4.4.11).

Proof. We may argue as in the proof of Proposition 4.4.5 to show that U^* satisfies (4.4.11). To prove the subsolution property, we assume the contrary that there exist $h \in C^{1,2}([0, T) \times \mathbb{R}^d)$ and $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ satisfying

$$0 = (U^* - h)(t_0, x_0) > (U^* - h)(t, x), \ \text{for any } (t, x) \in [0, T) \times \mathbb{R}^d, \ (t, x) \neq (t_0, x_0),$$

such that

$$\max\left\{c(t_0, x_0)h(t_0, x_0) - \frac{\partial h}{\partial t}(t_0, x_0) + H_*(\cdot, D_x h, D_x^2 h)(t_0, x_0), h(t_0, x_0) - g(x_0)\right\} > 0.$$

Since $U^*(t_0, x_0) = h(t_0, x_0)$ and $U \leq g$ by definition, continuity of g implies that $h(t_0, x_0) = U^*(t_0, x_0) \leq g(x_0)$. Therefore, the above inequality yields

(4.5.10) $\qquad c(t_0, x_0)h(t_0, x_0) - \frac{\partial h}{\partial t}(t_0, x_0) + H_*(\cdot, D_x h, D_x^2 h)(t_0, x_0) > 0.$

Define the function $\tilde{h}$ by

$$\tilde{h}(t, x) := h(t, x) + \varepsilon(|t - t_0|^2 + |x - x_0|^4).$$

Note that $(\tilde{h}, \partial_t \tilde{h}, D_x \tilde{h}, D_x^2 \tilde{h})(t_0, x_0) = (h, \partial_t h, D_x h, D_x^2 h)(t_0, x_0)$. Then, by the lower semicontinuity of H_*, there exists $r > 0$ with $t_0 + r < T$ such that

(4.5.11)
$$c(t, x)\tilde{h}(t, x) - \frac{\partial \tilde{h}}{\partial t}(t, x) + H^a(\cdot, D_x \tilde{h}, D_x^2 \tilde{h})(t, x) > 0, \ \forall \, a \in M \text{ and } (t, x) \in \bar{B}_r(t_0, x_0).$$

Now, define $\eta > 0$ by

(4.5.12)
$$\eta e^{\bar{c}T} := \min_{\partial B_r(t_0, x_0)} (\tilde{h} - h) > 0.$$

Take $(\hat{t}, \hat{x}) \in B_r(t_0, x_0)$ such that $|(U - \tilde{h})(\hat{t}, \hat{x})| < \eta/2$, and define $\pi \in \Pi_{\hat{t},T}$ by

(4.5.13)
$$\pi[\alpha] := \inf \left\{ s \geq \hat{t} \ \Big| \ (s, X_s^{\hat{t}, \hat{x}, \alpha}) \notin B_r(t_0, x_0) \right\}, \ \forall \alpha \in \mathcal{A}.$$

For any $\alpha \in \mathcal{A}_{\hat{t}}$, applying the product rule of stochastic calculus to $Y_s^{\hat{t}, \hat{x}, 1, \alpha} \tilde{h}(s, X_s^{\hat{t}, \hat{x}, \alpha})$, we get

$$\tilde{h}(\hat{t}, \hat{x}) = \mathbb{E}\Bigg[Y_{\pi[\alpha]}^{\hat{t}, \hat{x}, 1, \alpha} \tilde{h}(\pi[\alpha], X_{\pi[\alpha]}^{\hat{t}, \hat{x}, \alpha})$$
$$+ \int_{\hat{t}}^{\pi[\alpha]} Y_s^{\hat{t}, \hat{x}, 1, \alpha} \left(c\tilde{h} - \frac{\partial \tilde{h}}{\partial t} + H^{\alpha_s}(\cdot, D_x \tilde{h}, D_x^2 \tilde{h}) + f \right)(s, X_s^{\hat{t}, \hat{x}, \alpha}, \alpha_s) ds \Bigg]$$
$$> \mathbb{E}\Bigg[Y_{\pi[\alpha]}^{\hat{t}, \hat{x}, 1, \alpha} h(\pi[\alpha], X_{\pi[\alpha]}^{\hat{t}, \hat{x}, \alpha}) + \int_{\hat{t}}^{\pi[\alpha]} Y_s^{\hat{t}, \hat{x}, 1, \alpha} f(s, X_s^{\hat{t}, \hat{x}, \alpha}, \alpha_s) ds \Bigg] + \eta,$$

where the inequality follows from (4.5.12), (4.5.11) and $c \leq \bar{c}$. Moreover, by our choice of $(\hat{t}, \hat{x})$, we have $U(\hat{t}, \hat{x}) + \eta/2 > \tilde{h}(\hat{t}, \hat{x})$. It follows that

(4.5.14)
$$U(\hat{t}, \hat{x}) > \mathbb{E}\Bigg[Y_{\pi[\alpha]}^{\hat{t}, \hat{x}, 1, \alpha} h(\pi[\alpha], X_{\pi[\alpha]}^{\hat{t}, \hat{x}, \alpha}) + \int_{\hat{t}}^{\pi[\alpha]} Y_s^{\hat{t}, \hat{x}, 1, \alpha} f(s, X_s^{\hat{t}, \hat{x}, \alpha}, \alpha_s) ds \Bigg] + \frac{\eta}{2}, \ \forall \, \alpha \in \mathcal{A}_{\hat{t}}.$$

Finally, we conclude from the definition of U and Proposition 4.5.6 that there exist $\pi^* \in \Pi_{\hat{t},T}$ and $\hat{\alpha} \in \mathcal{A}_{\hat{t}}$ such that

$$U(\hat{t}, \hat{x}) = \bar{U}(\hat{t}, \hat{x}, 1, 0) \leq \sup_{\alpha \in \mathcal{A}_{\hat{t}}} \mathbb{E}\left[F\left(\mathbf{X}_{\pi^*[\alpha]}^{\hat{t}, \hat{x}, 1, 0, \alpha} \right) \right] \leq \mathbb{E}\left[F\left(\mathbf{X}_{\pi^*[\hat{\alpha}]}^{\hat{t}, \hat{x}, 1, 0, \hat{\alpha}} \right) \right] + \frac{\eta}{4}$$

$$\leq \mathbb{E}\left[Y_{\pi[\hat{\alpha}]}^{\hat{t}, \hat{x}, 1, \hat{\alpha}} h(\pi[\hat{\alpha}], X_{\pi[\hat{\alpha}]}^{\hat{t}, \hat{x}, \hat{\alpha}}) + Z_{\pi[\hat{\alpha}]}^{\hat{t}, \hat{x}, 1, 0, \hat{\alpha}} \right] + \frac{\eta}{2},$$

which contradicts (4.5.14). $\qquad \square$

4.6   Comparison

In this section, to state an appropriate comparison result, we assume a stronger version of (4.2.2) as follows: there exists $K > 0$ such that for any $t, s \in [0, T]$, $x, y \in \mathbb{R}^d$, and $u \in M$,

$$(4.6.1) \qquad |b(t, x, u) - b(s, y, u)| + |\sigma(t, x, u) - \sigma(s, y, u)| \le K(|t - s| + |x - y|).$$

Moreover, we impose an additional condition on f:

$$(4.6.2) \qquad f(t, x, u) \text{ is uniformly continuous in } (t, x), \text{ uniformly in } u \in M.$$

Note that the conditions (4.6.1) and (4.6.2), together with the linear growth condition (4.2.3) on b and σ, imply that the operator H defined in (4.3.13) is continuous, and $\overline{H} = H = H_*$.

Proposition 4.6.1. *Assume (4.6.1) and (4.6.2). Let u (resp. v) be an upper semicontinuous viscosity subsolution (resp. a lower semicontinuous viscosity supersolution), with polynomial growth in x, to*

$$(4.6.3) \quad \max\left\{ c(t, x)w - \frac{\partial w}{\partial t} + H(t, x, D_x w, D_x^2 w), \; w - g(x) \right\} = 0 \text{ on } [0, T) \times \mathbb{R}^d,$$

and $u(T, x) \le v(T, x)$ for all $x \in \mathbb{R}^d$. Then $u \le v$ on $[0, T) \times \mathbb{R}^d$.

Proof. For $\lambda > 0$, define $u^\lambda := e^{\lambda t} u(t, x)$, $v^\lambda := e^{\lambda t} v(t, x)$, and

$$H_\lambda(t, x, p, A) := \inf_{a \in M} \left\{ -b(t, x, a) \cdot p - \frac{1}{2} Tr[\sigma \sigma'(t, x, a)A] - e^{\lambda t} f(t, x, a) \right\}.$$

Note that the conditions (4.6.1) and (4.6.2), together with the linear growth condition (4.2.3) on b and σ and the polynomial growth condition (4.3.8) on f, imply that H_λ is continuous. By definition, u (resp. v) is upper semicontinuous (resp. lower semicontinuous) and has polynomial growth. Moreover, by direct calculations, the

subsolution property of u (resp. supersolution property of v) implies that u^λ (resp. v^λ) is a viscosity subsolution (resp. viscosiy supersolution) to

(4.6.4)
$$\max\left\{(c(t,x)+\lambda)\,w - \frac{\partial w}{\partial t} + H_\lambda(t,x,D_xw,D_x^2w),\ w - e^{\lambda t}g(x)\right\} = 0 \text{ on } [0,T)\times\mathbb{R}^d.$$

For any $(t,x,r,q,p,A) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{M}^d$, define

$$F_1(t,x,r,q,p,A) := (c(t,x)+\lambda)\,r - q + H_\lambda(t,x,p,A) \text{ and } F_2(t,x,r) := r - e^{\lambda t}g(x).$$

Since F_1 and F_2 are by definition continuous, so is $F_3 := \max\{F_1, F_2\}$. We can then write (4.6.4) as $F_3(t,x,w,\frac{\partial w}{\partial t}, D_xw, D_x^2w) = 0$.

From the polynomial growth condition on u^λ and v^λ, there is a $p > 0$ such that

$$\sup_{[0,T]\times\mathbb{R}^d} \frac{|u^\lambda(t,x)| + |v^\lambda(t,x)|}{1+|x|^p} < \infty.$$

Define $\gamma(x) := 1 + |x|^{2p}$ and set $\varphi(t,x) := e^{-\lambda t}\gamma(x)$. From the linear growth condition (4.2.3) on b and σ, a direct calculation shows that $|b(t,x,a)\cdot D_x\gamma + \frac{1}{2}Tr[\sigma\sigma'(t,x,a)D_x^2\gamma]| \le C\gamma(x)$ for some $C > 0$. It follows that

$$(c(t,x)+\lambda)\varphi - \frac{\partial\varphi}{\partial t} + \inf_{a\in M}\left\{-b(t,x,a)D_x\varphi - \frac{1}{2}Tr[\sigma\sigma'(t,x,a)D_x^2\varphi]\right\}$$
$$= e^{-\lambda t}\left([c(t,x)+2\lambda]\gamma + \inf_{a\in M}\left\{-b(t,x,a)D_x\gamma - \frac{1}{2}Tr[\sigma\sigma'(t,x,a)D_x^2\gamma]\right\}\right)$$
(4.6.5) $\quad \ge e^{-\lambda t}[c(t,x)+2\lambda - C]\gamma \ge 0,\ \text{if } \lambda \ge \frac{C}{2}.$

Now, take $\lambda \ge \frac{C}{2}$ and define $v_\varepsilon^\lambda := v^\lambda + \varepsilon\varphi$ for all $\varepsilon > 0$. By definition, v_ε^λ is lower semicontinuous. Given any $h \in C^{1,2}([0,T)\times\mathbb{R}^d)$ and $(t_0,x_0) \in [0,T)\times\mathbb{R}^d$ such that $v_\varepsilon^\lambda - h$ attains a local minimum, which equals 0, at (t_0,x_0), the supersolution property of v^λ implies either $F_1\left(\cdot, h(\cdot), \frac{\partial h}{\partial t}(\cdot), D_xh(\cdot), D_x^2h(\cdot)\right)(t_0,x_0) \ge 0$ or $F_2\left(\cdot, h(\cdot)\right)(t_0,x_0) \ge 0$. If the former holds true, we see from (4.6.5) that

$$F_1\left(\cdot, v_\varepsilon^\lambda(\cdot), \frac{\partial v_\varepsilon^\lambda}{\partial t}(\cdot), D_xv_\varepsilon^\lambda(\cdot), D_x^2v_\varepsilon^\lambda(\cdot)\right)(t_0,x_0) \ge 0;$$

if the latter holds true, then

$$F_2\left(\cdot, v_\varepsilon^\lambda(\cdot)\right)(t_0, x_0) = v_\varepsilon^\lambda(t_0, x_0) - e^{\lambda t_0}g(x_0) = F_2\left(\cdot, v^\lambda(\cdot)\right)(t_0, x_0) + \varepsilon\varphi(t_0, x_0)$$

$$= F_2\left(\cdot, h(\cdot)\right)(t_0, x_0) + \varepsilon\varphi(t_0, x_0) \geq 0.$$

Therefore, v_ε^λ is a lower semicontinuous viscosity supersolution to (4.6.4).

We would like to show $u^\lambda \leq v_\varepsilon^\lambda$ on $[0, T) \times \mathbb{R}^d$ for all $\varepsilon > 0$; then by sending ε to

0, we can conclude $u \leq v$ on $[0, T) \times \mathbb{R}^d$, as desired. We will argue by contradiction,

and thus assume that

$$N := \sup_{[0,T]\times\mathbb{R}^d} (u^\lambda - v_\varepsilon^\lambda)(t, x) > 0$$

From the polynomial growth condition on u^λ and v^λ and the definition of φ, we have

$$\lim_{|x|\to\infty} \sup_{[0,T]} (u^\lambda - v_\varepsilon^\lambda)(t, x) = -\infty.$$

It follows that there exists some bounded open set $\mathcal{O} \subset \mathbb{R}^d$ such that the maximum

N is attained at some point contained in $[0, T] \times \mathcal{O}$. For each $\delta > 0$, define

$$\Phi_\delta(t, s, x, y) := u^\lambda(t, x) - v_\varepsilon^\lambda(s, y) - \eta_\delta(t, s, x, y),$$

$$\text{with } \eta_\delta(t, s, x, y) := \frac{1}{2\delta}[|t - s|^2 + |x - y|^2].$$

Since Φ_δ is upper semicontinuous, it attains its maximum, denoted by N_δ, on the

compact set $[0, T]^2 \times \overline{\mathcal{O}}^2$ at some point $(t_\delta, s_\delta, x_\delta, y_\delta)$. Then, the upper semiconti-

nuity of $u^\lambda(t, x) - v_\varepsilon^\lambda(s, y)$ implies that $\left(u^\lambda(t_\delta, x_\delta) - v_\varepsilon^\lambda(s_\delta, y_\delta)\right)_\delta$ is bounded above;

moreover, it is also bounded below as

$$(4.6.6)\quad N \leq N_\delta = u^\lambda(t_\delta, x_\delta) - v_\varepsilon^\lambda(s_\delta, y_\delta) - \eta_\delta(t_\delta, s_\delta, x_\delta, y_\delta) \leq u^\lambda(t_\delta, x_\delta) - v_\varepsilon^\lambda(s_\delta, y_\delta).$$

Then we deduce from (4.6.6) and the boundedness of $\left(u^\lambda(t_\delta, x_\delta) - v_\varepsilon^\lambda(s_\delta, y_\delta)\right)_\delta$ the

boundedness of $(\eta_\delta(t_\delta, s_\delta, x_\delta, y_\delta))_\delta$. Note that the bounded sequence $(t_\delta, s_\delta, x_\delta, y_\delta)_\delta$

converges, up to a subsequence, to some point $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y}) \in [0,T]^2 \times \overline{\mathcal{O}}^2$. Then the definition of η_δ and the boundedness of $(\eta_\delta(t_\delta, s_\delta, x_\delta, y_\delta))_\delta$ imply that $\tilde{t} = \tilde{s}$ and $\tilde{x} = \tilde{y}$. Then, by sending δ to 0 in (4.6.6), we see that the last expression becomes $(u^\lambda - v_\varepsilon^\lambda)(\tilde{t}, \tilde{x}) \le N$, which implies that

$$(4.6.7) \qquad N_\delta \to N \text{ and } \eta_\delta(t_\delta, s_\delta, x_\delta, y_\delta) \to 0.$$

In view of Ishii's Lemma (see e.g. [91, Lemma 4.4.6]) and [91, Remark 4.4.9], for each $\delta > 0$, there exist $A_\delta, B_\delta \in \mathbb{M}^d$ such that

$$(4.6.8) \qquad Tr(CC'A_\delta - DD'B_\delta) \le \frac{3}{\delta}|C - D|^2 \text{ for all } C, D \in \mathbb{M}^d,$$

and

$$\left(\frac{1}{\delta}(t_\delta - s_\delta), \frac{1}{\delta}(x_\delta - y_\delta), A_\delta \right) \in \bar{\mathcal{P}}^{2,+} u^\lambda(t_\delta, x_\delta),$$

$$\left(\frac{1}{\delta}(t_\delta - s_\delta), \frac{1}{\delta}(x_\delta - y_\delta), B_\delta \right) \in \bar{\mathcal{P}}^{2,-} v_\varepsilon^\lambda(s_\delta, y_\delta),$$

where $\bar{\mathcal{P}}^{2,+} w(t,x)$ (resp. $\bar{\mathcal{P}}^{2,-} w(t,x)$) denotes the superjet (resp. subjet) of an upper semicontinuous (resp. a lower semicontinuous) function w at $(t,x) \in [0,T] \times \mathbb{R}^d$; for the definition of these notions, see e.g. [30] and [91]. Since the function $F_3 = \max\{F_1, F_2\}$ is continuous, we may apply [91, Lemma 4.4.5] and obtain that

$$\max \Big\{ \left(c(t_\delta, x_\delta) + \lambda \right) u^\lambda(t_\delta, x_\delta) - \frac{1}{\delta}(t_\delta - s_\delta) + H_\lambda(t_\delta, x_\delta, \frac{1}{\delta}(x_\delta - y_\delta), A_\delta),$$

$$u^\lambda(t_\delta, x_\delta) - e^{\lambda t_\delta} g(x_\delta) \Big\} \le 0,$$

$$\max \Big\{ \left(c(s_\delta, y_\delta) + \lambda \right) v_\varepsilon^\lambda(s_\delta, y_\delta) - \frac{1}{\delta}(t_\delta - s_\delta) + H_\lambda(s_\delta, y_\delta, \frac{1}{\delta}(x_\delta - y_\delta), B_\delta),$$

$$v_\varepsilon^\lambda(s_\delta, y_\delta) - e^{\lambda s_\delta} g(y_\delta) \Big\} \ge 0.$$

Noting that $\max\{a, b\} - \max\{c, d\} \ge \min\{a - c, b - d\}$ for any $a, b, c, d \in \mathbb{R}$, we get

$$\min \Big\{ \left(c(t_\delta, x_\delta) + \lambda \right) u^\lambda(t_\delta, x_\delta) - \left(c(s_\delta, y_\delta) + \lambda \right) v_\varepsilon^\lambda(s_\delta, y_\delta) + H_\lambda(t_\delta, x_\delta, \frac{1}{\delta}(x_\delta - y_\delta), A_\delta)$$

$$- H_\lambda(s_\delta, y_\delta, \frac{1}{\delta}(x_\delta - y_\delta), B_\delta), \; u^\lambda(t_\delta, x_\delta) - v_\varepsilon^\lambda(s_\delta, y_\delta) + e^{\lambda s_\delta} g(y_\delta) - e^{\lambda t_\delta} g(x_\delta) \Big\} \le 0.$$

Since $u^\lambda(t_\delta, x_\delta) - v_\varepsilon^\lambda(s_\delta, y_\delta) + e^{\lambda s_\delta} g(y_\delta) - e^{\lambda t_\delta} g(x_\delta) = N_\delta + \eta_\delta(t_\delta, s_\delta, x_\delta, y_\delta) + e^{\lambda s_\delta} g(y_\delta) - e^{\lambda t_\delta} g(x_\delta) \to N > 0$, we conclude from the previous inequality that as δ small enough,

$$(c(t_\delta, x_\delta) + \lambda) u^\lambda(t_\delta, x_\delta) - (c(s_\delta, y_\delta) + \lambda) v_\varepsilon^\lambda(s_\delta, y_\delta)$$

$$\leq H_\lambda(s_\delta, y_\delta, \frac{1}{\delta}(x_\delta - y_\delta), B_\delta) - H_\lambda(t_\delta, x_\delta, \frac{1}{\delta}(x_\delta - y_\delta), A_\delta)$$

$$\leq \mu(|t_\delta - s_\delta| + |x_\delta - y_\delta| + \frac{3}{\delta}|x_\delta - y_\delta|^2),$$

for some function μ such that $\mu(z) \to 0$ as $z \to 0$; note that the second inequality follows from (4.6.1), (4.6.2), and (4.6.8). Finally, by sending δ to 0 and using (4.6.7), we get $(c(\tilde{t}, \tilde{x}) + \lambda)N \leq 0$, a contradiction. $\qquad\square$

Now, we turn to the behavior of V_*, the lower semicontinuous envelope of V defined as in (4.1.3), at terminal time T.

Lemma 4.6.2. *For all $x \in \mathbb{R}^d$, $V_*(T, x) \geq g(x)$.*

Proof. Fix $\alpha \in \mathcal{A}$. Take an arbitrary sequence $(t_m, x_m) \to (T, x)$ with $t_m < T$ for all $m \in \mathbb{N}$. By the definition of V, we can choose for each $m \in \mathbb{N}$ a stopping time $\tau_m \in \mathcal{T}_{t_m, T}^{t_m}$ such that

$$V(t_m, x_m) \geq \inf_{\tau \in \mathcal{T}_{t_m, T}^{t_m}} \mathbb{E}\left[\int_{t_m}^\tau Y^{t_m, x_m, 1, \alpha} f(s, X^{t_m, x_m, \alpha}, \alpha_s) ds + Y_\tau^{t_m, x_m, 1, \alpha} g(X_\tau^{t_m, x_m, \alpha})\right]$$

$$\geq \mathbb{E}\left[\int_{t_m}^{\tau_m} Y^{t_m, x_m, 1, \alpha} f(s, X^{t_m, x_m, \alpha}, \alpha_s) ds + Y_{\tau_m}^{t_m, x_m, 1, \alpha} g(X_{\tau_m}^{t_m, x_m, \alpha})\right] - \frac{1}{m}.$$

Note that $\tau_m \to T$ as $\tau_m \in \mathcal{T}_{t_m, T}^{t_m}$ and $t_m \to T$. Then it follows from Fatou's lemma that $\liminf_{m \to \infty} V(t_m, x_m) \geq g(x)$. Since (t_m, x_m) is arbitrarily chosen, we conclude $V_*(T, x) \geq g(x)$. $\qquad\square$

Theorem 4.6.3. *Assume (4.6.1) and (4.6.2). Then $U^* = V$ on $[0, T] \times \mathbb{R}^d$. In particular, $U = V$ on $[0, T] \times \mathbb{R}^d$, i.e. the game has a value, which is the unique viscosity solution to (4.4.10) with terminal condition $w(T, x) = g(x)$ for $x \in \mathbb{R}^d$.*

Proof. Since by definition $U(t,x) \leq g(x)$ on $[0,T] \times \mathbb{R}^d$, we have $U^*(t,x) \leq g(x)$ on $[0,T] \times \mathbb{R}^d$ by the continuity of g. Then by Lemma 4.6.2 and the fact that $U^* \geq U \geq V$, we have $U^*(T,x) = V(T,x) = g(x)$ for all $x \in \mathbb{R}^d$. Recall that under (4.6.1) and (4.6.2), the function H is continuous, and $\overline{H} = H = H_*$. Now, in view of Propositions 4.4.5 and 4.5.7, and the fact that $U^*(T,\cdot) = V(T,\cdot)$ and $\overline{H} = H = H_*$, we conclude from Proposition 4.6.1 that $U^* = V$ on $[0,T] \times \mathbb{R}^d$, which in particular implies $U = V$ on $[0,T] \times \mathbb{R}^d$. $\qquad\square$

APPENDICES

APPENDIX A

Continuous Selection Results for Proposition 3.3.8

The goal of this subsection is to state and prove Proposition A.9, which is used in the proof of Proposition 3.3.8. Before we do that, we need some preparations concerning the theory of continuous selection in [85] and [25].

Definition A.4. Let X be a topological space.

(i) We say X is a T_1 space if for any distinct points $x, y \in X$, there exist open sets U_x and U_y such that U_x contains x but not y, and U_y contains y but not x.

(ii) We say X is a T_2(Hausdorff) space if for any distinct points $x, y \in X$, there exist open sets U_x and U_y such that $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$.

(iii) We say X is a paracompact space if for any collection $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ of open sets in X such that $\bigcup_{\alpha \in \mathcal{A}} X_\alpha = X$, there exists a collection $\{X_\beta\}_{\beta \in \mathcal{B}}$ of open sets in X satisfying

   (1) each X_β is a subset of some X_α;

   (2) $\bigcup_{\beta \in \mathcal{B}} X_\beta = X$;

   (3) given $x \in X$, there exists an open neighborhood of x which intersects only finitely many elements in $\{X_\beta\}_{\beta \in \mathcal{B}}$.

Definition A.5. Let X, Y be topological spaces. A set-valued map $\phi : X \mapsto 2^Y$ is

lower semicontinuous if, whenever $V \subset Y$ is open in Y, the set $\{x \in X \mid \phi(x) \cap V \neq \emptyset\}$ is open in X.

The main theorem in [85], Theorem 3.2″, gives the following result for continuous selection.

Proposition A.6. *Let X be a T_1 paracompact space, Y be a Banach space, and $\phi : X \mapsto 2^Y$ be a set-valued map such that $\phi(x)$ is a closed convex subset of Y for each $x \in X$. Then, if ϕ is lower semicontinuous, there exists a continuous function $f : X \mapsto Y$ such that $f(x) \in \phi(x)$ for all $x \in X$.*

Since the lower semicontinuity of ϕ can be difficult to prove in general, one may wonder whether there is a weaker condition sufficient for continuous selection. Brown [25] worked towards this direction, and characterized the weakest possible condition (it is therefore sufficient and necessary). For the special case where X is a Hausdorff paracompact space and Y is a real linear space with finite dimension n^*, given a set-valued map $\phi : X \mapsto 2^Y$, a sequence $\{\phi^{(n)}\}_{n \in \mathbb{N}}$ of set-valued maps was introduced in [25] via the following iteration:

$$\phi^{(1)}(x) := \{y \in \phi(x) \mid \text{Given } V \text{ open in } Y \text{ s.t. } y \in V, \text{ there is a}$$

(A.0.1) $$\text{neighborhood } U \text{ of } x \text{ s.t. } \forall x' \in U, \ \exists \ y' \in \phi(x') \cap V\};$$

$$\phi^{(n)}(x) := (\phi^{(n-1)})^{(1)}(x), \quad \text{for } n \geq 2.$$

The following result, taken from [25, Theorem 4.3], characterizes the possibility of continuous selection using $\phi^{(n^*)}$.

Proposition A.7. *Let X be a Hausdorff paracompact space, Y be a real linear space with finite dimension n^*, and $\phi : X \mapsto 2^Y$ be a set-valued map such that $\phi(x)$ is a closed convex subset of Y for each $x \in X$. Then, there exists a continuous function*

$f : X \mapsto Y$ such that $f(x) \in \phi(x)$ for all $x \in X$ if and only if $\phi^{(n^*)}(x) \neq \emptyset$ for all $x \in X$.

For the application in Chapter III, we would like to take $X = \bar{D}$ and $Y = \mathbb{S}^d$, where $D \subset E$ is a smooth bounded domain. Note that $\bar{D}$ is Hausdorff and paracompact as it is a metric space in $\mathbb{R}^d$ (see e.g. [76, Corollary 5.35]), and $\mathbb{S}^d$ is a real linear space with dimension $n^* := d(d+1)/2$. Fix two continuous functions $\gamma, \Gamma : E \mapsto (0, \infty)$ with $\gamma \leq \Gamma$, we consider the operator $F_{\gamma, \Gamma} : E \times \mathbb{S}^d \mapsto \mathbb{R}$ defined by

$$(A.0.2) \qquad F_{\gamma,\Gamma}(x, M) := \frac{1}{2} \mathcal{M}^+_{\gamma(x),\Gamma(x)}(M) = \frac{1}{2} \sup_{A \in \mathcal{A}(\gamma(x),\Gamma(x))} \mathrm{Tr}(AM).$$

Observe that $F_{\gamma,\Gamma}$ also satisfies (3.3.11)-(3.3.14), and in particular $F_{\theta,\Theta} = F$. Given $m \in \mathbb{N}$, we intend to show that there exists a continuous function $c_m : \bar{D} \mapsto \mathbb{S}^d$ such that for all $x \in \bar{D}$, $c_m(x) \in \mathcal{A}(\gamma(x), \Gamma(x))$ and $F_{\gamma,\Gamma}(x, D^2\eta_D) \leq L^{c_m(\cdot)}\eta_D(x) + 1/m$, with η_D given in Lemma 3.3.1. Note that since $\eta_D \in C^2(\bar{D})$ by Proposition 3.3.5, $D^2\eta_D$ is well-defined on ∂D. Also, see Proposition 3.3.8 for the purpose of finding such a function c_m. We then define the set-valued map $\varphi : D \mapsto \mathbb{S}^d$ by

$$\varphi(x) := \left\{ M \in \mathbb{S}^d \mid M \in \mathcal{A}(\gamma(x), \Gamma(x)) \text{ and } F_{\gamma,\Gamma}(x, D^2\eta_D) \leq L^M \eta_D(x) + 1/m \right\}.$$

For any $x \in \bar{D}$, we see from the definition of $F_{\gamma,\Gamma}$ that $\varphi(x) \neq \emptyset$. Moreover, $\varphi(x)$ is by definition a closed convex subset of $\mathbb{S}^d$. Then, we define $\varphi^{(n)}$ inductively as in (A.0.1) for all $n \in \mathbb{N}$. In view of Proposition A.7, such a function c_m exists if $\varphi^{(n^*)}(x) \neq \emptyset$ for all $x \in \bar{D}$. We claim that this is true. Actually, we will prove a stronger result in the next lemma: given $x \in \bar{D}$, $\varphi^{(n)}(x) \neq \emptyset$ for all $n \in \mathbb{N}$.

Recall that $B_\delta(x)$ denotes the open ball in $\mathbb{R}^d$ centered at $x \in \mathbb{R}^d$ with radius $\delta > 0$. In the following, we will denote by $B_\delta^{\bar{D}}(x)$ the corresponding open ball in $\bar{D}$ under the relative topology, i.e. $B_\delta^{\bar{D}}(x) := B_\delta(x) \cap \bar{D}$. Similarly, we will denote by

$B_\delta^{\mathbb{S}^d}(M)$ the corresponding open ball in $\mathbb{S}^d$ under the topology induced by $\|\cdot\|$ in (3.2.1).

Lemma A.8. *Fix a smooth bounded domain $D \subset E$, two continuous functions $\gamma, \Gamma : E \mapsto (0, \infty)$ with $\gamma \leq \Gamma$, and $m \in \mathbb{N}$. Let η_D be given as in Lemma 3.3.1. Then, given $x \in \bar{D}$, if $M \in \varphi(x)$ satisfies*

(A.0.3) $$F_{\gamma,\Gamma}(x, D^2\eta_D) < L^M\eta_D(x) + 1/m,$$

then $M \in \varphi^{(n)}(x)$ for all $n \in \mathbb{N}$.

Proof. Fix $M \in \varphi(x)$ such that (A.0.3) holds. We will first show that $M \in \varphi^{(1)}(x)$, and then complete the proof by an induction argument. Take $0 \leq \zeta < 1/m$ such that $F_{\gamma,\Gamma}(x, D^2\eta_D) = L^M\eta_D(x) + \zeta$. Set $\nu := 1/m - \zeta > 0$. Recall that $\eta_D \in C^2(\bar{D})$ from Proposition 3.3.5. By the continuity of the maps $x \mapsto F_{\gamma,\Gamma}(x, D^2\eta_D(x))$ (thanks to (3.3.14)) and $x \mapsto L^M\eta_D(x)$, we can take $\delta_1 > 0$ small enough such that the following holds for any $x' \in B_{\delta_1}^{\bar{D}}(x)$:

(A.0.4) $$F_{\gamma,\Gamma}(x', D^2\eta_D) < F_{\gamma,\Gamma}(x, D^2\eta_D) + \frac{\nu}{3} = L^M\eta_D(x) + \zeta + \frac{\nu}{3} < L^M\eta_D(x') + \zeta + \frac{2\nu}{3}.$$

Since $\|\cdot\|_{\max} \leq \|\cdot\|$, the map $(M, y) \mapsto L^M\eta_D(y)$ is continuous in M, uniformly in $y \in \bar{D}$. It follows that there exists $\beta > 0$ such that

(A.0.5) $$\|N - M\| < \beta \implies |L^N\eta_D(y) - L^M\eta_D(y)| < \frac{\nu}{3} \text{ for all } y \in \bar{D}.$$

Now, by the continuity of γ and Γ on $\bar{D}$, we can take $\delta_2 > 0$ such that $\max\{|\gamma(x') - \gamma(x)|, |\Gamma(x') - \Gamma(x)|\} < \beta$ for all $x' \in B_{\delta_2}^{\bar{D}}(x)$. For each $x' \in B_{\delta_2}^{\bar{D}}(x)$, we pick $M' \in \mathbb{S}^d$ satisfying

$$e_i(M') = \begin{cases} \gamma(x') & \text{if } e_i(M) < \gamma(x') \\ e_i(M) & \text{if } e_i(M) \in [\gamma(x'), \Gamma(x')], \\ \Gamma(x') & \text{if } e_i(M) > \Gamma(x'). \end{cases}$$

By construction, $M' \in \mathcal{A}(\gamma(x'), \Gamma(x'))$ and $\|M' - M\| \le \max\{|\gamma(x') - \gamma(x)|, |\Gamma(x') - \Gamma(x)|\} < \beta$, which implies

(A.0.6) $$|L^{M'}\eta_D(y) - L^M\eta_D(y)| < \frac{\nu}{3}, \text{ for all } y \in \bar{D}.$$

Finally, set $U := B_\delta^{\bar{D}}(x)$ with $\delta := \delta_1 \wedge \delta_2$. Then by (A.0.4) and (A.0.6), for any $x' \in U$ there exists $M' \in B_\beta^{\mathbb{S}^d}(M)$ such that $M' \in \mathcal{A}(\gamma(x'), \Gamma(x'))$ and

(A.0.7) $$F_{\gamma,\Gamma}(x', D^2\eta_D) < L^{M'}\eta_D(x') + 1/m,$$

which shows that $M' \in \varphi(x')$. Given any open set V in $\mathbb{S}^d$ such that $M \in V$, since we may take $\beta > 0$ in (A.0.5) small enough such that $B_\beta^{\mathbb{S}^d}(M) \subset V$, we conclude that $M' \in V$ also. It follows that $M \in \varphi^{(1)}(x)$.

Notice that what we have proved is the following result: for any $x \in \bar{D}$, if $M \in \varphi(x)$ satisfies (A.0.3), then $M \in \varphi^{(1)}(x)$. Since $M' \in \varphi(x')$ satisfies (A.0.7), the above result immediately gives $M' \in \varphi^{(1)}(x')$. We then obtain a stronger result: for any $x \in \bar{D}$, if $M \in \varphi(x)$ satisfies (A.0.3), then $M \in \varphi^{(2)}(x)$. But this stronger result, when applied again to $M' \in \varphi(x')$ satisfying (A.0.7), gives $M' \in \varphi^{(2)}(x')$. We, therefore, obtain that: for any $x \in \bar{D}$, if $M \in \varphi(x)$ satisfies (A.0.3), then $M \in \varphi^{(3)}(x)$. We can then argue inductively to conclude that $M \in \varphi^{(n)}(x)$ for all $n \in \mathbb{N}$. $\qquad\square$

Proposition A.9. *Fix a smooth bounded domain $D \subset E$ and two continuous functions $\gamma, \Gamma : E \mapsto (0, \infty)$ with $\gamma \le \Gamma$. Let η_D be given as in Lemma 3.3.1. For any $m \in \mathbb{N}$, there exists a continuous function $c_m : \bar{D} \mapsto \mathbb{S}^d$ such that*

$$c_m(x) \in \mathcal{A}(\gamma(x), \Gamma(x)) \quad and \quad F_{\gamma,\Gamma}(x, D^2\eta_D) \le L^{c_m(\cdot)}\eta_D(x) + 1/m, \quad for \ all \ x \in \bar{D}.$$

Proof. Fix $m \in \mathbb{N}$. As explained before Lemma A.8, $\bar{D}$ is a Hausdorff paracompact space, $\mathbb{S}^d$ is a real linear space with dimension $n^* := d(d+1)/2$, and $\varphi(x)$ is a closed

convex subset of $\mathbb{S}^d$ for all $x \in \bar{D}$. For each $x \in \bar{D}$, by the definition of $F_{\gamma,\Gamma}$ in (A.0.2), we can always find some $M \in \varphi(x)$ satisfying (A.0.3). By Lemma A.8, this implies $\varphi^{(n)}(x) \neq \emptyset$ for all $n \in \mathbb{N}$. In particular, we have $\varphi^{(n^*)}(x) \neq \emptyset$ for all $x \in \bar{D}$. Then the desired result follows from Proposition A.7. □

APPENDIX B

Proof of Lemma 3.3.10 (ii)

B.1 Proof of (3.3.38)

Pick $x_0 \in D$ and $R_0 > 0$ such that $B_{R_0}(x_0) \subset D$. For any $0 < R < R_0$, define

$$v_n(x) := u_n(x_0 + Rx) \quad \text{and} \quad \bar{H}(x, M) := H(x_0 + Rx, M).$$

Then we deduce from (3.3.11) and (3.3.36) that for any $x \in B_{R_0/R}(0)$,

$$\bar{H}(x, D^2 v_n(x)) + R^2 \delta_n v_n(x) = H(x_0 + Rx, D^2 v_n(x)) + R^2 \delta_n v_n(x) = R^2 f_n(x_0 + Rx).$$

Since $\bar{H}(x, M)$ satisfies (3.3.35) in $B_{R_0/R}(0)$, we can apply the estimate (3.3.37) to v_n and get

$$\sup_{\bar{B}_R(x_0)} u_n = \sup_{\bar{B}_1(0)} v_n \leq C \left\{ \inf_{\bar{B}_1(0)} v_n + R^2 \|f_n\|_{\mathcal{L}^d(B_{R_0}(x_0))} \right\}$$
$$= C \left\{ \inf_{\bar{B}_R(x_0)} u_n + R^2 \|f_n\|_{\mathcal{L}^d(B_{R_0}(x_0))} \right\},$$

where $C > 0$ depends only on R_0, d, λ, Λ, $\sup_n \delta_n$.

B.2 Proof of the Hölder continuity

Now, fix a compact connected set $K \subset D$. Set $R_0 := \frac{1}{2} d(\partial K, \partial D) > 0$. By [83, Lemma 2], there exists some $k^* \in \mathbb{N}$ such that the set $K' := \{x \in \mathbb{R}^d \mid d(x, K) \leq$

$R_0\} \subset D$ has the following property: any two points in K' can be joined by a polygonal line of at most k^* segments which lie entirely in K'. Fix $x_0 \in K'$. By the definition of R_0, we have $B_{R_0}(x_0) \subset D$. For each $n \in \mathbb{N}$, we consider the nondecreasing function $w^n : (0, R_0] \mapsto \mathbb{R}$ defined by

$$w^n(R) := M_R^n - m_R^n, \quad \text{where} \quad M_R^n := \max_{\bar{B}_R(x_0)} u_n, \ m_R^n := \min_{\bar{B}_R(x_0)} u_n.$$

For each $R \in (0, R_0]$, we obtain from (3.3.36) that $\{u_n - m_R^n\}_{n \in \mathbb{N}}$ is sequence of nonnegative continuous viscosity solution to

$$H(x, D^2(u_n - m_R^n)) + \delta_n(u_n - m_R^n) = f_n - \delta_n m_R^n \ \text{ in } B_R(x_0).$$

By the estimate (3.3.38), there is some $C > 0$, independent of n and x_0, such that

(B.2.1)
$$M_{R/4}^n - m_R^n = \sup_{\bar{B}_{R/4}(x_0)} (u_n(x) - m_R^n) \leq C \inf_{\bar{B}_{R/4}(x_0)} (u_n(x) - m_R^n) + AR^2$$
$$= C(m_{R/4}^n - m_R^n) + AR^2,$$

where $A > 0$ is a constant depends on C and R_0, but not n (thanks to the uniform boundedness of $\{u_n\}_{n \in \mathbb{N}}$ and the boundedness of $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{L}^d(D)$). Define $\bar{H}(x, M) := -H(x, -M)$. Then we deduce again from (3.3.36) that $\{M_R^n - u_n\}_{n \in \mathbb{N}}$ is a sequence of nonnegative continuous viscosity solutions to

$$\bar{H}(x, D^2(M_R^n - u_n)) + \delta_n(M_R^n - u_n) = -H(x, D^2 u_n) + \delta_n(M_R^n - u_n)$$
$$= -f_n + \delta_n M_R^n \ \text{ in } B_R(x_0).$$

Observe that $\bar{H}$ also satisfies (3.3.11) and (3.3.35). Thus, we can apply the estimate (3.3.38) and get

(B.2.2)
$$M_R^n - m_{R/4}^n = \sup_{\bar{B}_{R/4}(x_0)} (M_R^n - u_n(x)) \leq C \inf_{\bar{B}_{R/4}(x_0)} (M_R^n - u_n(x)) + AR^2$$
$$= C(M_R^n - M_{R/4}^n) + AR^2,$$

where C and A are as above. Summing (B.2.1) and (B.2.2), we get

$$w^n(R/4) = M^n_{R/4} - m^n_{R/4} \leq \frac{C-1}{C+1}(M^n_R - m^n_R) + A'R^2 = \frac{C-1}{C+1}w^n(R) + A'R^2,$$

where $A' > 0$ depends on C and R_0, and is independent of R and n. By applying [53, Lemma 8.23] to the above inequality, for any $\beta \in (0,1)$, we can find some $\tilde{C} > 0$ (depending on C, R_0 and A', but not n) such that $w^n(R) \leq \tilde{C}R^\beta$, for all $R \leq R_0$. This implies the following result: for any $x, y \in K'$ with $|x - y| \leq R_0$, we can take $x_0 = x$ in the above analysis and obtain $|u_n(x) - u_n(y)| \leq w^n(|x - y|) \leq \tilde{C}|x - y|^\beta$ for all $n \in \mathbb{N}$. For the case where $|x - y| > R_0$, recall that x and y can be joined by a polygonal line of k segments which lie entirely in K', for some $k \leq k^*$. On the j-th segment, pick points $x^j_1, x^j_2, \cdots, x^j_{\ell_j}$ along the segment such that $x^j_1, x^j_{\ell_j}$ are the two endpoints, $|x^j_i - x^j_{i+1}| = R_0$ for $i = 1, \cdots, \ell_j - 2$, and $|x^j_{\ell_j - 1} - x^j_{\ell_j}| \leq R_0$. Since K' is bounded, there must be a uniform bound $\ell^* > 0$ such that $\ell_j \leq \ell^*$ for all j. Then, for all $n \in \mathbb{N}$, we have

$$|u_n(x) - u_n(y)| \leq \sum_{j=1}^{k} \sum_{i=1}^{\ell_j - 1} |u_n(x^j_i) - u_n(x^j_{i+1})| \leq \sum_{j=1}^{k} \sum_{i=1}^{\ell_j - 1} \tilde{C}|x^j_i - x^j_{i+1}|^\beta \leq k^* \ell^* \tilde{C}|x - y|^\beta.$$

APPENDIX C

Properties of Shifted Objects in the Space $C([0,T]; \mathbb{R}^d)$

This appendix is devoted to rigorous proofs for the properties of shifted objects stated in Propositions 4.2.7, 4.2.8, 4.2.9, and Lemma 4.3.10. To this end, we will first derive several auxiliary results.

Recall the definitions introduced in Subsection 4.2.1. Fix $t \in [0,T]$. For any $A \subseteq \Omega$, $\tilde{A} \subseteq \Omega^t$, and $x \in \mathbb{R}^d$, we set

$$\tilde{A}_x := \{\tilde{\omega} \in \tilde{A} \mid \tilde{\omega}_t = x\},$$

and define

$$A^{t,\omega} := \{\tilde{\omega} \in \Omega^t \mid \omega \otimes_t \tilde{\omega} \in A\}, \quad A_x^{t,\omega} := (A^{t,\omega})_x, \quad \omega \otimes_t \tilde{A} := \{\omega \otimes_t \tilde{\omega} \mid \tilde{\omega} \in \tilde{A}\}.$$

Given a random time $\tau : \Omega \mapsto [0,\infty]$, whenever $\omega \in \Omega$ is fixed, we simplify our notation as $A^{\tau,\omega} = A^{\tau(\omega),\omega}$. We also consider

(C.0.1) $$\mathcal{H}_s^t := \psi_t^{-1} \mathcal{G}_s^{t,0} \subseteq \mathcal{G}_s^t, \ \forall s \in [t,T].$$

Note that the inclusion follows from the Borel measurability of ψ_t. Finally, while $\mathbb{E}$ denotes the expectation taken under $\overline{\mathbb{P}}$, in this appendix we also consider $\mathbb{E}_{\mathbb{P}}$, the expectation taken under $\mathbb{P}$.

Lemma C.1. *Fix* $t \in [0,T]$ *and* $\omega \in \Omega$. *For any* $r \in [t,T]$, $A \in \mathcal{G}_r$, $\tilde{A} \in \mathcal{G}_r^t$, *and* $\xi \in L^0(\Omega, \mathcal{G}_r)$,

(i) $A_x^{t,\omega} = A_0^{t,\omega} + x$ and $A_x^{t,\omega} \in \mathcal{G}_r^{t,x}$, $\forall x \in \mathbb{R}^d$.

(ii) $A^{t,\omega} = \psi_t^{-1} A_0^{t,\omega} \in \mathcal{H}_r^t \subseteq \mathcal{G}_r^t$ and $\mathbb{P}^t(A^{t,\omega}) = \mathbb{P}^{t,x}(A_x^{t,\omega}) = \mathbb{P}^{t,x}(A^{t,\omega})$, $\forall x \in \mathbb{R}^d$.

(iii) $\phi_t^{-1} A^{t,\omega} \in \phi_t^{-1} \mathcal{H}_r^t \subseteq \mathcal{G}_r$ and $\mathbb{P}(\phi_t^{-1} A^{t,\omega}) = \mathbb{P}^t(A^{t,\omega})$.

(iv) $\omega \otimes_t \tilde{A}_{\omega_t} \in \mathcal{G}_r$. Hence, $\omega \otimes_t A_{\omega_t}^{t,\omega} \in \mathcal{G}_r$.

(v) For any Borel subset $\mathcal{E}$ of $\mathbb{R}$, $(\xi^{t,\omega})^{-1}(\mathcal{E}) \in \phi_t^{-1}\mathcal{H}_r^t \subseteq \mathcal{G}_r$. Hence, $\xi^{t,\omega} \in L^0(\Omega, \mathcal{G}_r)$.

Proof. (i) Fix $x \in \mathbb{R}^d$. Since $\tilde{\omega} \in A_0^{t,\omega} \Leftrightarrow \omega \otimes_t \tilde{\omega} \in A$ and $\tilde{\omega}_t = 0 \Leftrightarrow (\omega \otimes_t (\tilde{\omega} + x)). = \omega.1_{[0,t]}(\cdot) + ((\tilde{\omega}. + x) - (\tilde{\omega}_t + x) + \omega_t)1_{(t,T]}(\cdot) = (\omega \otimes_t \tilde{\omega}). \in A$ and $(\tilde{\omega} + x)_t = x \Leftrightarrow \tilde{\omega} + x \in A_x^{t,\omega}$, we conclude $A_x^{t,\omega} = A_0^{t,\omega} + x$.

Set $\Lambda := \{A \subseteq \Omega \mid A_x^{t,\omega} \in \mathcal{G}_r^{t,x}\}$. Note that $\Omega \in \Lambda$ since $\Omega_x^{t,\omega} = \{\tilde{w} \in \Omega^t \mid \omega \otimes_t \tilde{\omega} \in \Omega, \tilde{\omega}_t = x\} = (\Omega^t)_x \in \mathcal{G}_r^{t,x}$. Given $A \in \Lambda$, we have $(A^c)_x^{t,\omega} = (\Omega^t)_x \setminus \{\tilde{\omega} \in \Omega^t \mid \omega \otimes_t \tilde{\omega} \in A, \tilde{\omega}_t = x\} = (\Omega^t)_x \setminus A_x^{t,\omega} \in \mathcal{G}_r^{t,x}$, which shows $A^c \in \Lambda$. Given $\{A_i\}_{i\in\mathbb{N}} \subset \Lambda$, we have $\left(\bigcup_{i\in\mathbb{N}} A_i\right)_x^{t,\omega} = \bigcup_{i\in\mathbb{N}}\{\tilde{\omega} \in \Omega^t \mid \omega \otimes_t \tilde{\omega} \in A_i, \tilde{\omega}_t = x\} = \bigcup_{i\in\mathbb{N}}(A_i)_x^{t,\omega} \in \mathcal{G}_r^{t,x}$, which shows $\bigcup_{i\in\mathbb{N}} A_i \in \Lambda$. Thus, we conclude Λ is a σ-algebra of Ω. For any $x \in \mathbb{Q}^d$ and $\lambda \in \mathbb{Q}_+$, the set of positive rationals, let $O_\lambda(x)$ denote the open ball in $\mathbb{R}^d$ centered at x with radius λ. Note from [66, p.307] that for each $s \in [0, T]$, $\mathcal{G}_r^s$ is countably generated by

(C.0.2)
$$\mathcal{C}_r^s := \left\{ \bigcap_{i=1}^m (W_{t_i}^s)^{-1}(O_{\lambda_i}(x_i)) \,\middle|\, m \in \mathbb{N}, \ t_i \in \mathbb{Q}, \ s \le t_1 < \cdots < t_m \le r, \ x_i \in \mathbb{Q}^d, \ \lambda_i \in \mathbb{Q}_+ \right\}.$$

Given $C = \bigcap_{i=1}^m (W_{t_i})^{-1}(O_{\lambda_i}(x_i))$ in $\mathcal{C}_r = \mathcal{C}_r^0$, if $t_m \ge t$, set $k = \min\{i = 1, \cdots, m \mid t_i \ge t\}$; otherwise, set $k = m + 1$. Then, if $\omega_{t_i} \notin O_{\lambda_i}(x_i)$ for some $i = 1, \cdots, k - 1$, we have $C_x^{t,\omega} = \emptyset \in \mathcal{G}_r^{t,x}$; if $k = m + 1$ and $w_{t_i} \in O_{\lambda_i}(x_i) \ \forall i = 1, \cdots, m$, we have $C_x^{t,\omega} = (\Omega^t)_x \in \mathcal{G}_r^{t,x}$; for all other cases,

(C.0.3)
$$C_x^{t,\omega} = \{W_t^t = x\} \cap \bigcap_{i=k}^m (W_{t_i}^t)^{-1}\left(O_{\lambda_i}(x_i - \omega_t + x)\right) \in \mathcal{G}_r^{t,x}.$$

Thus, $\mathcal{C}_r \subseteq \Lambda$, which implies $\mathcal{G}_r = \sigma(\mathcal{C}_r) \subseteq \Lambda$. Now, for any $A \in \mathcal{G}_r$, $A_x^{t,\omega} \in \mathcal{G}_r^{t,x} \subseteq \mathcal{G}_r^t$.

(ii) Observe from part (i) that $\tilde{\omega} \in A^{t,\omega} \Leftrightarrow \tilde{\omega} \in A_{\tilde{\omega}_t}^{t,\omega} \Leftrightarrow \tilde{\omega} - \tilde{\omega}_t \in A_0^{t,\omega}$ i.e.

$\psi_t(\tilde{\omega}) \in A_0^{t,\omega} \Leftrightarrow \tilde{\omega} \in \psi_t^{-1}(A_0^{t,\omega})$. Thus, $A^{t,\omega} = \psi_t^{-1}(A_0^{t,\omega}) \in \psi_t^{-1}(\mathcal{G}_r^{t,0}) = \mathcal{H}_r^t \subseteq \mathcal{G}_r^t$,

thanks to part (i) and (C.0.1). Then, using part (i) again, $\mathbb{P}^t(A^{t,\omega}) = \mathbb{P}^t(A_0^{t,\omega}) = \mathbb{P}^{t,x}(A_0^{t,\omega} + x) = \mathbb{P}^{t,x}(A_x^{t,\omega}) = \mathbb{P}^{t,x}(A^{t,\omega})$, $\forall x \in \mathbb{R}^d$.

(iii) By part (ii) and the Borel measurability of $\phi_t : (\Omega, \mathcal{G}_r) \mapsto (\Omega^t, \mathcal{G}_r^t)$, we immediately have $\phi_t^{-1} A^{t,\omega} \in \phi_t^{-1} \mathcal{H}_r^t \subseteq \mathcal{G}_r$. Now, by property (e") in [66, p.84] and part (ii), $\mathbb{P}[\phi_t^{-1} A^{t,\omega} \mid \mathcal{G}_{t+}](\omega') = \mathbb{P}^{t,\omega_t'}(A^{t,\omega}) = \mathbb{P}^t(A^{t,\omega})$ for $\mathbb{P}$-a.e. $\omega' \in \Omega$, which implies $\mathbb{P}[\phi_t^{-1} A^{t,\omega}] = \mathbb{P}^t(A^{t,\omega})$.

(iv) Set $\Lambda := \{\tilde{A} \subseteq \Omega^t \mid \omega \otimes_t \tilde{A}_{\omega_t} \in \mathcal{G}_r\}$. Let $\mathcal{C}_r^t$ be given as in (C.0.2). For any $C = \bigcap_{i=1}^m (W_{t_i}^t)^{-1}(O_{\lambda_i}(x_i))$ in $\mathcal{C}_r^t$, we deduce from the continuity of paths in Ω that

$$\omega \otimes_t C_{\omega_t} = \{\omega' \in \Omega \mid \omega_s' = \omega_s \; \forall s \in \mathbb{Q} \cap [0, t) \text{ and } \omega_{t_i}' \in O_{\lambda_i}(x_i) \text{ for } i = 1, \cdots, m\}$$

$$= \left(\bigcap_{s \in \mathbb{Q} \cap [0,t)} (W_s)^{-1}(\omega_s) \right) \cap \left(\bigcap_{i=1}^m (W_{t_i})^{-1}(O_{\lambda_i}(x_i)) \right) \in \mathcal{G}_r.$$

Thus, we have $\mathcal{C}_r^t \subseteq \Lambda$. Given $\{\tilde{A}_i\}_{i \in \mathbb{N}} \subset \Lambda$, we have $\omega \otimes_t (\bigcup_{i \in \mathbb{N}} \tilde{A}_i)_{\omega_t} = \bigcup_{i \in \mathbb{N}} (\omega \otimes_t (\tilde{A}_i)_{\omega_t}) \in \mathcal{G}_r$, which shows $\bigcup_{i \in \mathbb{N}} \tilde{A}_i \in \Lambda$; this in particular implies

$$\Omega^t = \bigcup_{n \in \mathbb{N}} (W_r^t)^{-1}(O_n(0)) \in \Lambda.$$

Given $\tilde{A} \in \Lambda$, we have $\omega \otimes_t (\tilde{A}^c)_{\omega_t} = (\omega \otimes_t (\Omega^t)_{\omega_t}) \setminus (\omega \otimes_t \tilde{A}_{\omega_t}) \in \mathcal{G}_r$, which shows $\tilde{A}^c \in \Lambda$. Hence, Λ is a σ-algebra of Ω^t, which implies $\mathcal{G}_r^t = \sigma(\mathcal{C}_r^t) \subseteq \Lambda$. Now, by part (i), we must have $\omega \otimes_t A_{\omega_t}^{t,\omega} \in \mathcal{G}_r$.

(v) Since $\xi^{-1}(\mathcal{E}) \in \mathcal{G}_r$, $(\xi^{t,\omega})^{-1}(\mathcal{E}) = \{\omega' \in \Omega \mid \xi(\omega \otimes_t \phi_t(\omega')) \in \mathcal{E}\} = \{\omega' \in \Omega \mid \omega \otimes_t \phi_t(\omega') \in \xi^{-1}(\mathcal{E})\} = \phi_t^{-1}(\xi^{-1}(\mathcal{E}))^{t,\omega} \in \phi_t^{-1} \mathcal{H}_r^t \subseteq \mathcal{G}_r$, thanks to part (iii). $\square$

In light of Theorem 1.3.4 and equation (1.3.15) in [105], for any $\mathbb{G}$-stopping time τ, there exists a family $\{Q_\tau^\omega\}_{\omega \in \Omega}$ of probability measures on $(\Omega, \mathcal{G}_T)$, called a regular conditional probability distribution (r.c.p.d.) of $\mathbb{P}$ given $\mathcal{G}_\tau$, such that

(i) for each $A \in \mathcal{G}_T$, the mapping $\omega \mapsto Q_\tau^\omega(A)$ is $\mathcal{G}_\tau$-measurable.

(ii) for each $A \in \mathcal{G}_T$, it holds for $\mathbb{P}$-a.e. $\omega \in \Omega$ that $\mathbb{P}[A \mid \mathcal{G}_\tau](\omega) = Q_\tau^\omega(A)$.

(iii) for each $\omega \in \Omega$, $Q_\tau^\omega \left(\omega \otimes_\tau (\Omega^{\tau(\omega)})_{\omega_\tau} \right) = 1$.

By property (iii) above and Lemma C.1 (iv), for any fixed $\omega \in \Omega$, we can define a probability measure $Q^{\tau,\omega}$ on $\left(\Omega^{\tau(\omega)}, \mathcal{G}_T^{\tau(\omega)} \right)$ by

$$Q^{\tau,\omega}(\tilde{A}) := Q_\tau^\omega(\omega \otimes_\tau \tilde{A}_{\omega_\tau}), \ \forall \tilde{A} \in \mathcal{G}_T^{\tau(\omega)}.$$

Then, combining properties (ii) and (iii) above, we have: for $A \in \mathcal{G}_T$, it holds for $\mathbb{P}$-a.e. $\omega \in \Omega$ that

$$(C.0.4) \quad \mathbb{P}[A \mid \mathcal{G}_\tau](\omega) = Q_\tau^\omega \left((\omega \otimes_\tau (\Omega^{\tau(\omega)})_{\omega_\tau}) \cap A \right) = Q_\tau^\omega(\omega \otimes_\tau A_{\omega_\tau}^{\tau,\omega}) = Q^{\tau,\omega}(A^{\tau,\omega}).$$

Note that the r.c.p.d. $\{Q_\tau^\omega\}_{\omega \in \Omega}$ is generally not unique. For each $(t,x) \in [0,T] \times \mathbb{R}^d$, observe that the shifted Wiener measure $\mathbb{P}^{t,x}$ can be characterized as the unique solution to the martingale problem for the operator $L := \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j}$ starting from time t with initial value x (see [104, Remark 7.1.23] and [105, Exercise 6.7.3]). Then, thanks to the strong Markov property of solutions to the martingale problem (see e.g. [105, Theorem 6.2.2]), there exists a particular r.c.p.d. $\{Q_\tau^\omega\}_{\omega \in \Omega}$ such that $Q^{\tau,\omega} = \mathbb{P}^{\tau(\omega),\omega_{\tau(\omega)}}$. Now, by (C.0.4) and Lemma C.1 (ii), we have: for $A \in \mathcal{G}_T$,

$$(C.0.5) \qquad \mathbb{P}[A \mid \mathcal{G}_\tau](\omega) = \mathbb{P}^{\tau(\omega),\omega_{\tau(\omega)}}(A^{\tau,\omega}) = \mathbb{P}^{\tau(\omega)}(A^{\tau,\omega}), \ \mathbb{P}\text{-a.s.}$$

So far, we have restricted ourselves to $\mathbb{G}$-stopping times. We say a random variable $\tau : \Omega \mapsto [0,\infty]$ is a $\mathbb{G}$-optional time if $\{\tau < t\} \in \mathcal{G}_t$ for all $t \in [0,T]$. In the following, we obtain a generalized version of (C.0.5) for $\mathbb{G}$-optional times.

Lemma C.2. *Fix a $\mathbb{G}$-optional time $\tau \leq T$. For any $A \in \mathcal{G}_T$,*

$$\mathbb{P}[A \mid \mathcal{G}_{\tau+}](\omega) = \mathbb{P}^{\tau(\omega)}(A^{\tau,\omega}) \ \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Proof. **Step 1:** By [66, Problem 1.2.24], we can take a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of $\mathbb{G}$-stopping times such that $\tau_n(\omega) \downarrow \tau(\omega)$ for all $\omega \in \Omega$. Fix $A \in \mathcal{G}_T$. For each $n \in \mathbb{N}$, (C.0.5) implies that for any $B \in \mathcal{G}_{\tau_n}$,

$$(C.0.6) \qquad \mathbb{E}_{\mathbb{P}}[1_A 1_B] = \mathbb{E}_{\mathbb{P}}[\mathbb{P}^{\tau_n(\omega)}(A^{\tau_n,\omega})1_B].$$

Then, for any $B \in \mathcal{G}_{\tau+}$, we must have (C.0.6) for all $n \in \mathbb{N}$, since $\mathcal{G}_{\tau+} = \bigcap_{n\in\mathbb{N}} \mathcal{G}_{\tau_n}$. Now, by taking the limit in n and assuming that for each $\omega \in \Omega$

$$(C.0.7) \qquad \lim_{n\to\infty} \mathbb{P}^{\tau_n(\omega)}(A^{\tau_n,\omega}) = \mathbb{P}^{\tau(\omega)}(A^{\tau,\omega}),$$

we obtain from the dominated convergence theorem that $\mathbb{E}_{\mathbb{P}}[1_A 1_B] = \mathbb{E}_{\mathbb{P}}[\mathbb{P}^{\tau(\omega)}(A^{\tau,\omega})1_B]$. Since $B \in \mathcal{G}_{\tau+}$ is arbitrary, we conclude $\mathbb{P}[A \mid \mathcal{G}_{\tau+}](\omega) = \mathbb{P}^{\tau(\omega)}(A^{\tau,\omega})$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Step 2: It remains to prove (C.0.7). Fix $\omega \in \Omega$ and set $\Lambda := \{A \subseteq \Omega \mid$ (C.0.7) holds$\}$. Since $\Omega^{s,\omega} = \Omega^s$, $\forall s \in [0,T]$, (C.0.7) holds for Ω and thus $\Omega \in \Lambda$. Given $A \in \Lambda$, we have $\mathbb{P}^{\tau_n(\omega)}[(A^c)^{\tau_n,\omega}] = \mathbb{P}^{\tau_n(\omega)}[(A^{\tau_n,\omega})^c] = 1 - \mathbb{P}^{\tau_n(\omega)}(A^{\tau_n,\omega}) \to 1 - \mathbb{P}^{\tau(\omega)}(A^{\tau,\omega}) = \mathbb{P}^{\tau(\omega)}[(A^{\tau,\omega})^c] = \mathbb{P}^{\tau(\omega)}[(A^c)^{\tau,\omega}]$, which shows $A^c \in \Lambda$. Given a sequence $\{A_i\}_{i\in\mathbb{N}}$ of disjoint sets in Λ, observe that $\{A_i^{s,\omega}\}_{i\in\mathbb{N}}$ is a sequence of disjoint sets in Ω^s for any $s \in [0,T]$. Then we have $\mathbb{P}^{\tau_n(\omega)}[(\bigcup_{i\in\mathbb{N}} A_i)^{\tau_n,\omega}] = \mathbb{P}^{\tau_n(\omega)}[\bigcup_{i\in\mathbb{N}} A_i^{\tau_n,\omega}] = \sum_{i\in\mathbb{N}} \mathbb{P}^{\tau_n(\omega)}(A_i^{\tau_n,\omega}) \to \sum_{i\in\mathbb{N}} \mathbb{P}^{\tau(\omega)}(A_i^{\tau,\omega}) = \mathbb{P}^{\tau(\omega)}[\bigcup_{i\in\mathbb{N}} A_i^{\tau,\omega}] = \mathbb{P}^{\tau(\omega)}[(\bigcup_{i\in\mathbb{N}} A_i)^{\tau,\omega}]$, which shows $\bigcup_{i\in\mathbb{N}} A_i \in \Lambda$. Thus, we conclude that Λ is a σ-algebra of Ω.

As mentioned in the proof of Lemma C.1 (i), $\mathcal{G}_T$ is countably generated by $\mathcal{C}_T = \mathcal{C}_T^0$ given in (C.0.2). Given $C = \bigcap_{i=1}^m (W_{t_i})^{-1}(O_{\lambda_i}(x_i))$ in $\mathcal{C}_T$, if $t_m \geq \tau(\omega)$ we set $k := \min\{i = 1, \cdots, m \mid t_i \geq \tau(\omega)\}$; otherwise, set $k := m+1$. We see that: **1.** If $\omega_{t_i} \notin O_{\lambda_i}(x_i)$ for some $i = 1, \cdots, k-1$, then $C^{s,\omega} = \emptyset$ $\forall s \in [\tau(\omega), T]$ and thus (C.0.7) holds for C. **2.** If $k = m+1$ and $\omega_{t_i} \in O(x_i)$ for all $i = 1, \cdots, m$, we have $C^{s,\omega} = \Omega^s$ $\forall s \in [\tau(\omega), T]$ and thus (C.0.7) still holds for C. **3.** For all other cases, $C_{\omega_s}^{s,\omega}$ is of the form in (C.0.3) $\forall s \in [\tau(\omega), T]$. Let B be a d-dimensional Brownian motion defined

on any given filtered probability space $(E, \mathcal{I}, \{\mathcal{I}_s\}_{s \geq 0}, P)$. Then by Lemma C.1 (ii),

$$\mathbb{P}^{\tau_n(\omega)}[C^{\tau_n, \omega}] = \mathbb{P}^{\tau_n(\omega), \omega_{\tau_n(\omega)}}[C^{\tau_n, \omega}_{\omega_\tau}] = P[B_{t_i - \tau_n(\omega)} \in O_{\lambda_i}(x_i - \omega_{\tau_n(\omega)}), i = k \cdots, m]$$

$$\to P[B_{t_i - \tau(\omega)} \in O_{\lambda_i}(x_i - \omega_{\tau(\omega)}), i = k \cdots, m] = \mathbb{P}^{\tau(\omega), \omega_{\tau(\omega)}}[C^{\tau, \omega}_{\omega_\tau}] = \mathbb{P}^{\tau(\omega)}[C^{\tau, \omega}].$$

Hence, we conclude that $\mathcal{C}_T \subseteq \Lambda$ and therefore $\mathcal{G}_T = \sigma(\mathcal{C}_T) \subseteq \Lambda$. □

Now, we want to generalize Lemma C.1 to incorporate $\mathbb{F}$-stopping times.

Lemma C.3. *Fix $\theta \in \mathcal{T}$. We have*

(i) For any $\overline{N} \in \overline{\mathcal{N}}$, $\overline{N}^{\theta, \omega} \in \overline{\mathcal{N}}^{\theta(\omega)}$ and $\phi_\theta^{-1} \overline{N}^{\theta, \omega} \in \overline{\mathcal{N}}$ for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$.

(ii) For any $r \in [0, T]$ and $A \in \mathcal{F}_r$, it holds for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$ that

$$\text{if } \theta(\omega) \leq r, \quad A^{\theta, \omega} \in \mathcal{H}_r^{\theta(\omega)} \cup \overline{\mathcal{N}}^{\theta(\omega)} \subseteq \overline{\mathcal{G}}_r^{\theta(\omega)} \quad \text{and} \quad \phi_\theta^{-1} A^{\theta, \omega} \in \mathcal{F}_r^{\theta(\omega)}.$$

(iii) For any $r \in [0, T]$ and $\xi \in L^0(\Omega, \mathcal{F}_r)$, it holds for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$ that

$$\text{if } \theta(\omega) \leq r, \quad \xi^{\theta, \omega} \in L^0(\Omega, \mathcal{F}_r^{\theta(\omega)}).$$

Proof. (i) Take $N \in \mathcal{N}$ such that $\overline{N} \subseteq N$. By [66, Exercise 2.7.11], there exists a $\mathbb{G}$-optional time τ such that $\overline{N}_1 := \{\theta \neq \tau\} \in \overline{\mathcal{N}}$. By Lemma C.2, there exists $\overline{N}_2 \in \overline{\mathcal{N}} \subset \overline{\mathcal{N}}$ such that $0 = \mathbb{P}[N \mid \mathcal{G}_{\tau+}](\omega) = \mathbb{P}^{\tau(\omega)}(N^{\tau, \omega})$, for $\omega \in \Omega \setminus \overline{N}_2$. Thus, for $\omega \in \Omega \setminus (\overline{N}_1 \cup \overline{N}_2)$, we have $0 = \mathbb{P}^{\tau(\omega)}(N^{\tau, \omega}) = \mathbb{P}^{\theta(\omega)}(N^{\theta, \omega})$, i.e. $N^{\theta, \omega} \in \mathcal{N}^{\theta(\omega)}$. Since $\overline{N}^{\theta, \omega} \subseteq N^{\theta, \omega}$, we have $\overline{N}^{\theta, \omega} \in \overline{\mathcal{N}}^{\theta(\omega)}$ $\overline{\mathbb{P}}$-a.s.

On the other hand, from Lemma C.1 (iii), $\mathbb{P}(\phi_\theta^{-1} N^{\theta, \omega}) = \mathbb{P}^{\theta(\omega)}(N^{\theta, \omega}) = 0$ for $\omega \in \Omega \setminus (\overline{N}_1 \cup \overline{N}_2)$, which shows $\phi_\theta^{-1} N^{\theta, \omega} \in \mathcal{N}$ $\overline{\mathbb{P}}$-a.s. Since $\phi_\theta^{-1} \overline{N}^{\theta, \omega} \subseteq \phi_\theta^{-1} N^{\theta, \omega}$, we conclude $\phi_\theta^{-1} \overline{N}^{\theta, \omega} \in \overline{\mathcal{N}}$ $\overline{\mathbb{P}}$-a.s.

(ii) By [66, Problem 2.7.3], there exist $\tilde{A} \in \mathcal{G}_r$ and $\overline{N} \in \overline{\mathcal{N}}$ such that $A = \tilde{A} \cup \overline{N}$ and $\tilde{A} \cap \overline{N} = \emptyset$. From Lemma C.1 (ii), we know that for any $\omega \in \Omega$, if $\theta(\omega) \leq r$ then $\tilde{A}^{\theta, \omega} \in \mathcal{H}_r^{\theta(\omega)} \subseteq \mathcal{G}_r^{\theta(\omega)}$. Also, from part (i) we have $\overline{N}^{\theta, \omega} \in \overline{\mathcal{N}}^{\theta(\omega)}$ $\overline{\mathbb{P}}$-a.s. We

therefore conclude that for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$, if $\theta(\omega) \leq r$, then $A^{\theta,\omega} = \tilde{A}^{\theta,\omega} \cup \overline{N}^{\theta,\omega} \in \mathcal{H}_r^{\theta(\omega)} \cup \overline{\mathcal{N}}^{\theta(\omega)} \subseteq \overline{\mathcal{G}}_r^{\theta(\omega)}$. Then, thanks to part (i) and Definition 4.2.1, it holds $\overline{\mathbb{P}}$-a.s. that $\phi_\theta^{-1} A^{\theta,\omega} = \phi_\theta^{-1} \tilde{A}^{\theta,\omega} \cup \phi_\theta^{-1} \overline{N}^{\theta,\omega} \in \phi_\theta^{-1} \mathcal{H}_r^{\theta(\omega)} \cup \overline{\mathcal{N}} \subseteq \mathcal{F}_r^{\theta(\omega)}$ if $\theta(\omega) \leq r$.

(iii) Let $\mathcal{E}$ be a Borel subset of $\mathbb{R}$. Since $\xi^{-1}(\mathcal{E}) \in \mathcal{F}_r$, we see from part (ii) that, for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$, $\left(\xi^{\theta,\omega}\right)^{-1}(\mathcal{E}) = \{\omega' \in \Omega \mid \xi(\omega \otimes_\theta \phi_\theta(\omega')) \in \mathcal{E}\} = \{\omega' \in \Omega \mid \omega \otimes_\theta \phi_\theta(\omega') \in \xi^{-1}(\mathcal{E})\} = \phi_\theta^{-1}(\xi^{-1}(\mathcal{E}))^{\theta,\omega} \in \mathcal{F}_r^{\theta(\omega)}$ if $\theta(\omega) \leq r$. $\qquad \square$

Now, we generalize Lemma C.2 to incorporate $\mathbb{F}$-stopping times.

Lemma C.4. *Fix $\theta \in \mathcal{T}$. For any $A \in \mathcal{F}_T$, $\overline{\mathbb{P}}[A \mid \mathcal{F}_\theta](\omega) = \overline{\mathbb{P}}^{\theta(\omega)}(A^{\theta,\omega})$, for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$.*

Proof. Thanks again to [66, Exercise 2.7.11], we may take a $\mathbb{G}$-optional time τ such that $\overline{N}_1 := \{\theta \neq \tau\} \in \overline{\mathcal{N}}$ and $\mathcal{F}_\tau = \mathcal{F}_\theta$. Moreover, we have $A = \tilde{A} \cup \overline{N}$ for some $\tilde{A} \in \mathcal{G}_T$ and $\overline{N} \in \overline{\mathcal{N}}$ with $\tilde{A} \cap \overline{N} = \emptyset$, by using [66, Exercise 2.7.3]. Then, in view of Lemma C.1 (ii), Lemma C.3 (i), and Lemma C.2, we can take some $\overline{N}_2 \in \overline{\mathcal{N}}$ such that for $\omega \in \Omega \setminus (\overline{N}_1 \cup \overline{N}_2)$,

$$\overline{\mathbb{P}}^{\theta(\omega)}(A^{\theta,\omega}) = \overline{\mathbb{P}}^{\tau(\omega)}(A^{\tau,\omega}) = \overline{\mathbb{P}}^{\tau(\omega)}(\tilde{A}^{\tau,\omega}) + \overline{\mathbb{P}}^{\tau(\omega)}(\overline{N}^{\tau,\omega}) = \mathbb{P}^{\tau(\omega)}(\tilde{A}^{\tau,\omega})$$

(C.0.8)

$$= \mathbb{P}[\tilde{A} \mid \mathcal{G}_{\tau+}](\omega) = \overline{\mathbb{P}}[\tilde{A} \mid \mathcal{G}_{\tau+}](\omega) = \overline{\mathbb{P}}[A \mid \mathcal{G}_{\tau+}](\omega).$$

For any $B \in \mathcal{F}_\tau$, $B = \tilde{B} \cup \overline{N}'$ for some $\tilde{B} \in \mathcal{G}_\tau \subseteq \mathcal{G}_{\tau+}$ and $\overline{N}' \in \overline{\mathcal{N}}$ with $\tilde{B} \cap \overline{N}' = \emptyset$, thanks again to [66, Exercise 2.7.3]. We then deduce from (C.0.8) that $\mathbb{E}[1_{\tilde{A}} 1_B] = \mathbb{E}[1_{\tilde{A}} 1_{\tilde{B}}] = \mathbb{E}\left[\overline{\mathbb{P}}^{\theta(\omega)}(A^{\theta,\omega}) 1_{\tilde{B}}\right] = \mathbb{E}\left[\overline{\mathbb{P}}^{\theta(\omega)}(A^{\theta,\omega}) 1_B\right]$. Hence, we conclude $\overline{\mathbb{P}}^{\theta(\omega)}(A^{\theta,\omega}) = \overline{\mathbb{P}}[A \mid \mathcal{F}_\tau](\omega) = \overline{\mathbb{P}}[A \mid \mathcal{F}_\theta](\omega)$, for $\omega \in \Omega \setminus (\overline{N}_1 \cup \overline{N}_2)$. $\qquad \square$

Finally, we are able to generalize Lemma C.1 (iii) to incorporate $\mathbb{F}$-stopping times.

Proposition C.5. *Fix $\theta \in \mathcal{T}$. We have*

(i) for any $A \in \mathcal{F}_T$, $\overline{\mathbb{P}}[A \mid \mathcal{F}_\theta](\omega) = \overline{\mathbb{P}}[\phi_\theta^{-1} A^{\theta,\omega}]$, for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$.

(ii) for any $\xi \in L^1(\Omega, \mathcal{F}_T, \overline{\mathbb{P}})$, $\mathbb{E}[\xi \mid \mathcal{F}_\theta](\omega) = \mathbb{E}\left[\xi^{\theta,\omega}\right]$ for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$.

Proof. (i) By Lemma C.3 (i) and Lemma C.1 (iii), it holds $\overline{\mathbb{P}}$-a.s. that

$$
\begin{aligned}
\overline{\mathbb{P}}[\phi_\theta^{-1} A^{\theta,\omega}] &= \overline{\mathbb{P}}[\phi_\theta^{-1} \tilde{A}^{\theta,\omega}] + \overline{\mathbb{P}}[\phi_\theta^{-1} \overline{N}^{\theta,\omega}] = \mathbb{P}[\phi_\theta^{-1} \tilde{A}^{\theta,\omega}] = \mathbb{P}^{\theta(\omega)}[\tilde{A}^{\theta,\omega}] \\
&= \overline{\mathbb{P}}^{\theta(\omega)}[\tilde{A}^{\theta,\omega}] = \overline{\mathbb{P}}^{\theta(\omega)}[A^{\theta,\omega}].
\end{aligned}
$$

The desired result then follows from the above equality and Lemma C.4.

(ii) Given $A \in \mathcal{F}_T$, observe that for any fixed $\omega \in \Omega$,

$$
(1_A)^{\theta,\omega}(\omega') = 1_A\left(\omega \otimes_\theta \phi_\theta(\omega')\right) = 1_{\phi_\theta^{-1} A^{\theta,\omega}}(\omega').
$$

Then we see immediately from part (i) that part (ii) is true for $\xi = 1_A$. It follows that part (ii) also holds true for any $\mathcal{F}_T$-measurable simple function ξ. For any positive $\xi \in L^1(\Omega, \mathcal{F}_T, \overline{\mathbb{P}})$, we can take a sequence $\{\xi_n\}_{n \in \mathbb{N}}$ of $\mathcal{F}_T$-measurable simple functions such that $\xi_n(\omega) \uparrow \xi(\omega)$ $\forall \omega \in \Omega$. By the monotone convergence theorem, there exists $\overline{N} \in \overline{\mathcal{N}}$ such that $\mathbb{E}[\xi_n \mid \mathcal{F}_\theta](\omega) \uparrow \mathbb{E}[\xi \mid \mathcal{F}_\theta](\omega)$, for $\omega \in \Omega \setminus \overline{N}$. For each $n \in \mathbb{N}$, since ξ_n is an $\mathcal{F}_T$-measurable simple function, there exists $\overline{N}_n \in \overline{\mathcal{N}}$ such that $\mathbb{E}[\xi_n \mid \mathcal{F}_\theta](\omega) = \mathbb{E}\left[(\xi_n)^{\theta,\omega}\right]$, for $\omega \in \Omega \setminus \overline{N}_n$. Finally, noting that there exists $\overline{N}' \in \overline{\mathcal{N}}$ such that $\xi^{\theta,\omega}$ is $\mathcal{F}_T$-measurable for $\omega \in \Omega \setminus \overline{N}'$ (from Lemma C.3 (iii)) and that $(\xi_n)^{\theta,\omega}(\omega') \uparrow \xi^{\theta,\omega}(\omega')$ $\forall \omega' \in \Omega$ (from the everywhere convergence $\xi_n \uparrow \xi$), we obtain from the monotone convergence theorem again that for $\omega \in \Omega \setminus \left(\left(\bigcup_{n \in \mathbb{N}} \overline{N}_n\right) \cup \overline{N} \cup \overline{N}'\right)$,

$$
\mathbb{E}[\xi \mid \mathcal{F}_\theta](\omega) = \lim_{n \to \infty} \mathbb{E}[\xi_n \mid \mathcal{F}_\theta](\omega) = \lim_{n \to \infty} \mathbb{E}[(\xi_n)^{\theta,\omega}] = \mathbb{E}[\xi^{\theta,\omega}].
$$

The same result holds true for any general $\xi \in L^1(\Omega, \mathcal{F}_T, \overline{\mathbb{P}})$ as $\xi = \xi^+ - \xi^-$. $\qquad\square$

C.1   Proof of Proposition 4.2.7

Proof. (i) Set $\Lambda := \{A \subseteq \Omega \mid \overline{\mathbb{P}}(A \cap B) = \overline{\mathbb{P}}(A)\overline{\mathbb{P}}(B) \ \forall B \in \mathcal{F}_t\}$. It can be checked that Λ is a σ-algebra of Ω. Take $A \in \phi_t^{-1} \mathcal{H}_T^t \cup \overline{\mathcal{N}}$. If $A \in \overline{\mathcal{N}}$, it is trivial that $A \in \Lambda$;

if $A = \phi_t^{-1}C$ with $C \in \mathcal{H}_T^t$, then for any $B \in \mathcal{F}_t$,

$$\overline{\mathbb{P}}(A \cap B) = \overline{\mathbb{P}}(B \cap \phi_t^{-1}C) = \mathbb{E}\left[\overline{\mathbb{P}}(B \cap \phi_t^{-1}C \mid \mathcal{F}_t)\right] = \mathbb{E}\left[\overline{\mathbb{P}}(B \cap \phi_t^{-1}C \mid \mathcal{F}_t)(\omega)1_B(\omega)\right].$$

By Proposition C.5 (i), for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$, $\overline{\mathbb{P}}(B \cap \phi_t^{-1}C \mid \mathcal{F}_t)(\omega) = \overline{\mathbb{P}}[\phi_t^{-1}(B \cap \phi_t^{-1}C)^{t,\omega}] = \overline{\mathbb{P}}[\phi_t^{-1}C] = \overline{\mathbb{P}}(A)$ if $\omega \in B$. We therefore have $\overline{\mathbb{P}}(A \cap B) = \overline{\mathbb{P}}(A)\overline{\mathbb{P}}(B)$, and conclude $A \in \Lambda$. It follows that $\phi_t^{-1}\mathcal{H}_T^t \cup \overline{\mathcal{N}} \subseteq \Lambda$, which implies $\mathcal{F}_T^t = \sigma(\phi_t^{-1}\mathcal{H}_T^t \cup \overline{\mathcal{N}}) \subseteq \Lambda$. Thus, $\mathcal{F}_T^t$ and $\mathcal{F}_t$ are independent.

(ii) Let Δ denote the set operation of symmetric difference. Set $\Lambda := \{A \subseteq \Omega \mid (\phi_t^{-1}A^{t,\omega})\Delta A \in \overline{\mathcal{N}}$ for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega\}$. It can be checked that Λ is a σ-algebra of Ω. Take $A \in \phi_t^{-1}\mathcal{H}_T^t \cup \overline{\mathcal{N}}$. If $A \in \overline{\mathcal{N}}$, we see from Lemma C.3 (i) that $A \in \Lambda$; if $A = \phi_t^{-1}C$ with $C \in \mathcal{H}_T^t$, then $\phi_t^{-1}A^{t,\omega} = \phi_t^{-1}C = A$ for all $\omega \in \Omega$, and thus $A \in \Lambda$. We then conclude that $\mathcal{F}_T^t = \sigma(\phi_t^{-1}\mathcal{H}_T^t \cup \overline{\mathcal{N}}) \subseteq \Lambda$.

Take a sequence $\{\xi_n\}$ of random variables in $L^0(\Omega, \mathcal{F}_T^t)$ taking countably many values $\{r_i\}_{i \in \mathbb{N}}$ such that $\xi_n(\omega) \to \xi(\omega)$ for all $\omega \in \Omega$. This everywhere convergence implies that for any fixed $\omega \in \Omega$, $(\xi_n)^{t,\omega}(\omega') \to \xi^{t,\omega}(\omega')$ for all $\omega' \in \Omega$. Now, fix $n \in \mathbb{N}$. For each $i \in \mathbb{N}$, since $(\xi_n)^{-1}\{r_i\} \in \mathcal{F}_T^t \subseteq \Lambda$, there exists $\overline{N}_i^n \in \overline{\mathcal{N}}$ such that for $\omega \in \Omega \setminus \overline{N}_i^n$,

(C.1.1)
$$\left(\left[(\xi_n)^{t,\omega}\right]^{-1}\{r_i\}\right)\Delta(\xi_n)^{-1}\{r_i\}$$
$$= \left[\phi_t^{-1}\left((\xi_n)^{-1}\{r_i\}\right)^{t,\omega}\right]\Delta(\xi_n)^{-1}\{r_i\} =: \overline{M}_i^n \in \overline{\mathcal{N}},$$

where the first equality follows from the calculation in the proof of Lemma C.3 (iii). Then, we deduce from (C.1.1) that: for any fixed $\omega \in \Omega \setminus \bigcup_{i \in \mathbb{N}} \overline{N}_i^n$, $(\xi_n)^{t,\omega}(\omega') = \xi_n(\omega')$ for all $\omega' \in \Omega \setminus \bigcup_{i \in \mathbb{N}} \overline{M}_i^n$. It follows that: for any fixed $\omega \in \Omega \setminus \bigcup_{i,n \in \mathbb{N}} \overline{N}_i^n$, $(\xi_n)^{t,\omega}(\omega') = \xi_n(\omega')$ for all $\omega' \in \Omega \setminus \bigcup_{i,n \in \mathbb{N}} \overline{M}_i^n$ and $n \in \mathbb{N}$. Setting $\overline{N} = \bigcup_{i,n \in \mathbb{N}} \overline{N}_i^n$ and $\overline{M} = \bigcup_{i,n \in \mathbb{N}} \overline{M}_i^n$, we obtain that for any $\omega \in \Omega \setminus \overline{N}$,

$$\xi(\omega') = \lim_{n \to \infty} \xi_n(\omega') = \lim_{n \to \infty} (\xi_n)^{t,\omega}(\omega') = \xi^{t,\omega}(\omega'), \text{ for } \omega' \in \Omega \setminus \overline{M}.$$

□

C.2 Proof of Proposition 4.2.8

Proof. Take a sequence of stopping times $\{\tau_i\}_{i\in\mathbb{N}} \subset \mathcal{T}$ such that τ_i takes values in $\{m/2^i \mid m \in \mathbb{N}\}$ for each $i \in \mathbb{N}$ and $\tau_i(\omega) \downarrow \tau(\omega)$ for all $\omega \in \Omega$ (thanks to [66, Problem 1.2.24]). Set $\overline{N} := \{\tau < \theta\} \in \overline{\mathcal{N}}$. Since $\tau_i(\omega) \downarrow \tau(\omega)$ for all $\omega \in \Omega$, we have $\tau_i \geq \theta$ on $\Omega \setminus \overline{N}$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $r_m^i := m/2^i$, $m \in \mathbb{N}$. Since $\{\tau_i \leq r_m^i\} \in \mathcal{F}_{r_m^i}$ for all $m \in \mathbb{N}$, we deduce from Lemma C.3 (ii) and the countability of $\{r_m^i\}_{m\in\mathbb{N}}$ that there exists $\overline{N}^i \in \overline{\mathcal{N}}$ such that for $\omega \in \Omega \setminus \overline{N}^i$,

(C.2.1)   if $\theta(\omega) \leq r_m^i$, $\phi_\theta^{-1}\{\tau_i \leq r_m^i\}^{\theta,\omega} \in \mathcal{F}_{r_m^i}^{\theta(\omega)}$ for all $m \in \mathbb{N}$.

Fix $r \in [0,T]$. For any $\omega \in \Omega \setminus (\overline{N} \cup \overline{N}^i)$, if $\theta(\omega) > r$, then $\tau_i(\omega) \geq \theta(\omega) > r$ and thus $\phi_\theta^{-1}\{\tau_i \leq r\}^{\theta,\omega} = \phi_\theta^{-1}\emptyset = \emptyset \in \mathcal{F}_r^{\theta(\omega)}$; if $\theta(\omega) \leq r$, there are two cases: **1.** $\exists$ $m^* \in \mathbb{N}$ s.t. $r_{m^*}^i \in [\theta(\omega), r]$ and $r_{m^*+1}^i > r$. Then, by (C.2.1), $\phi_\theta^{-1}\{\tau_i \leq r\}^{\theta,\omega} = \phi_\theta^{-1}\{\tau_i \leq r_{m^*}^i\}^{\theta,\omega} \in \mathcal{F}_{r_{m^*}^i}^{\theta(\omega)} \subset \mathcal{F}_r^{\theta(\omega)}$; **2.** $\exists$ $m^* \in \mathbb{N}$ s.t. $r_{m^*}^i < \theta(\omega)$ and $r_{m^*+1}^i > r$. Since $\tau_i(\omega) \geq \theta(\omega) > r_{m^*}^i$, $\phi_\theta^{-1}\{\tau_i \leq r\}^{\theta,\omega} = \phi_\theta^{-1}\{\tau_i \leq r_{m^*}^i\}^{\theta,\omega} = \phi_\theta^{-1}\emptyset = \emptyset \in \mathcal{F}_r^{\theta(\omega)}$. Thus, for $\omega \in \Omega \setminus (\overline{N} \cup \overline{N}^i)$, we have $\phi_\theta^{-1}\{\tau_i \leq r\}^{\theta,\omega} \in \mathcal{F}_r^{\theta(\omega)}$, and therefore

$$\{\tau_i^{\theta,\omega} \leq r\} = \{\tau_i(\omega \otimes_\theta \phi_\theta(\omega')) \leq r\} = \phi_\theta^{-1}\{\tau_i \leq r\}^{\theta,\omega} \in \mathcal{F}_r^{\theta(\omega)}, \ \forall\, r \in [0,T].$$

This shows that $\tau_i^{\theta,\omega} \in \mathcal{T}_{\theta(\omega),T}^{\theta(\omega)}$ for $\omega \in \Omega \setminus (\overline{N} \cup \overline{N}^i)$. Hence, for $\omega \in \Omega \setminus \left(\overline{N} \cup (\bigcup_{i\in\mathbb{N}} \overline{N}^i)\right)$, we have $\tau_i^{\theta,\omega} \in \mathcal{T}_{\theta(\omega),T}^{\theta(\omega)}$ $\forall i \in \mathbb{N}$. Finally, since the filtration $\mathbb{F}^{\theta(\omega)}$ is right-continuous, $\tau^{\theta,\omega}(\omega') = \downarrow \lim_{i\to\infty} \tau_i^{\theta,\omega}(\omega')$ (this is true since $\tau_i \downarrow \tau$ everywhere) must also be a stopping time in $\mathcal{T}_{\theta(\omega),T}^{\theta(\omega)}$. □

C.3 Proof of Proposition 4.2.9

Recall the metric $\tilde{\rho}$ on $\mathcal{A}$ defined in (4.2.8). We say $\beta \in \mathcal{A}$ is a step control if there exists a subdivision $0 = t_0 < t_1 < \cdots < t_m = T$, $m \in \mathbb{N}$, of the interval $[0,T]$

such that $\beta_t = \beta_{t_i}$ for $t \in [t_i, t_{i+1})$ for $i = 0, 1, \cdots, m-1$.

Proof. By [77, Lemma 3.2.6], there exist a sequence $\{\alpha^n\}$ of step controls such that $\alpha^n \to \alpha$. For each $n \in \mathbb{N}$, in view of Proposition 4.2.7 (ii), there exist $\overline{N}_n, \overline{M}_n \in \overline{\mathcal{N}}$ such that: for any fixed $\omega \in \Omega \backslash \overline{N}_n$, $(\alpha_r^n)^{t,\omega}(\omega') = \alpha_r^n(\omega')$ for $(r, \omega') \in [0, T] \times (\Omega \backslash \overline{M}_n)$. It follows that: for any fixed $\omega \in \Omega \setminus \bigcup_{n \in \mathbb{N}} \overline{N}_n$, $(\alpha_r^n)^{t,\omega}(\omega') = \alpha_r^n(\omega')$ for all $(r, \omega') \in [0, T] \times (\Omega \setminus \bigcup_{n \in \mathbb{N}} \overline{M}_n)$ and $n \in \mathbb{N}$. With the aid of Proposition C.5 (ii), we obtain

$$
\begin{aligned}
0 = \lim_{n \to \infty} \tilde{\rho}(\alpha^n, \alpha) &= \lim_{n \to \infty} \mathbb{E}\left[\int_0^T \rho'(\alpha_r^n, \alpha_r) dr\right] \\
&= \lim_{n \to \infty} \mathbb{E}\left(\mathbb{E}\left[\int_0^T \rho'(\alpha_r^n, \alpha_r) dr \,\middle|\, \mathcal{F}_t\right](\omega)\right) \\
&= \lim_{n \to \infty} \int \int \left(\int_0^T \rho'(\alpha_r^n, \alpha_r) dr\right)^{t,\omega}(\omega') \, d\overline{\mathbb{P}}(\omega') \, d\overline{\mathbb{P}}(\omega) \\
&= \lim_{n \to \infty} \int \int \int_0^T \rho'\left((\alpha_r^n)^{t,\omega}(\omega'), \alpha_r^{t,\omega}(\omega')\right) dr \, d\overline{\mathbb{P}}(\omega') \, d\overline{\mathbb{P}}(\omega) \\
&= \lim_{n \to \infty} \int \tilde{\rho}\left((\alpha^n)^{t,\omega}, \alpha^{t,\omega}\right) d\overline{\mathbb{P}}(\omega) = \lim_{n \to \infty} \int \tilde{\rho}(\alpha^n, \alpha^{t,\omega}) d\overline{\mathbb{P}}(\omega) \\
&= \int \lim_{n \to \infty} \tilde{\rho}(\alpha^n, \alpha^{t,\omega}) d\overline{\mathbb{P}}(\omega),
\end{aligned}
$$

where the last equality is due to the dominated convergence theorem. This implies that $0 = \lim_{n \to \infty} \tilde{\rho}(\alpha^n, \alpha^{t,\omega})$, for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$. Recalling that $\alpha^n \to \alpha$, we conclude that $\tilde{\rho}(\alpha^{t,\omega}, \alpha) = 0$ for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$. The second assertion follows immediately from [77, Exercise 3.2.4]. $\qquad\square$

C.4 Proof of Lemma 4.3.10

Proof. By taking $\xi = F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha})$ in Proposition C.5 (ii) and using Remark 4.2.6 (ii),

$$
\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha}) \mid \mathcal{F}_\theta](\omega) = \mathbb{E}\left[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha})^{\theta,\omega}\right] = \int F\left(\mathbf{X}_\tau^{t,\mathbf{x},\alpha}\left(\omega \otimes_\theta \phi_\theta(\omega')\right)\right) d\overline{\mathbb{P}}(\omega')
$$

$$
= \int F\left(\mathbf{X}_{\tau^{\theta,\omega}}^{\theta(\omega), \mathbf{X}_\theta^{t,\mathbf{x},\alpha}(\omega), \alpha^{\theta,\omega}}(\omega')\right) d\overline{\mathbb{P}}(\omega') = J\left(\theta(\omega), \mathbf{X}_\theta^{t,\mathbf{x},\alpha}(\omega); \alpha^{\theta,\omega}, \tau^{\theta,\omega}\right),
$$

for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$. $\qquad\square$

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