SELF-DISCOVERY MODULE (GUI) FOR SINGULAR VECTORS: THE "GREATEST STRETCH" METHOD FOR 2×2 MATRICES

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1. Introduction

Modern linear algebra textbooks include coverage of the singular value decomposition, or SVD for short. The utility of the SVD is extremely widespread - see for example [2], [3], [4], [5]. This note describes an interactive module that provides a visualization technique in which the user enters her/his own 2×2 matrix and then visually self-discovers the singular vectors and values of the matrix.

Definition 1. *The* singular value decomposition *of a $m \times n$ matrix A of rank r has the form $A = UDV^T$ where U and V are square orthogonal matrices and D is an $m \times n$ diagonal matrix. If U and V have columns $\mathbf{u}_1, \mathbf{u}_2, \ldots$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots$, respectively, then $\mathbf{u}_i$ is the i^{th}* left singular vector*, and $\mathbf{v}_i$ is the i^{th}* right singular vector*. The nonzero diagonal entries of D are called the* singular values *and satisfy $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_r > 0$.*

2. The self-discovery GUI module

One starts by downloading the attached GUI: "Self-Discovery of Singular Vectors and Values." The self-discovery visualization method prompts users to input a 2×2 matrix A of their choice by filling in the four matrix entries a, b, c and d, and then to click on the t-slidebar. The value of t regulates the angle of inclination of the unit vector (x, y) on the left-hand coordinate system of the display. The user explores by click/sliding along the slidebar to discover where A takes various (x, y) points on the unit circle.

Real 2×2 matrices map the unit circle onto ellipses. The t-value that gives the point on the ellipse that is farthest from the origin is the crucial one. This t-value is important because the vector in this direction is $\sigma_1 \mathbf{u}_1$ where σ_1 is a scalar and $\mathbf{u}_1$ is a unit vector. For this particular t, the left singular vector $\mathbf{u}_1$ is the normalized version of the vector (u, v) displayed in the right coordinate system, and the length $|(u, v)|$ is the singular value σ_1. Furthermore, the right singular vector $\mathbf{v}_1$ is the vector (x, y) on the left hand coordinate system that is the preimage of $\sigma_1 \mathbf{u}_1$.

3. The details

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be considered as a mapping of points in the xy-coordinate system to points in the uv-coordinate system via

$$\begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

If $x = \cos(t)$ and $y = \sin(t)$, then $u = ax + by$ and $v = cx + dy$.
The length of $(x, y) = |(x, y)| = \sqrt{x^2 + y^2} = \sqrt{\cos^2(t) + \sin^2(t)} = 1$, and
the length of $(u, v) = |(u, v)| = \sqrt{u^2 + v^2} = \sqrt{(a\cos(t) + b\sin(t))^2 + (c\cos(t) + d\sin(t))^2}$.
The normalized (u, v) is the vector $\frac{1}{|(u,v)|}(u, v)$.

4. Second singular vectors and value

To recapitulate, the first left singular vector $\mathbf{u}_1$ is the unit vector that lies in the direction of greatest stretch, and the first singular value σ_1 is the length of this longest semi-axis. Furthermore, the first right singular vector $\mathbf{v}_1$ is the unit vector that is the preimage of $\sigma_1 \mathbf{u}_1$. Together these constitute a useful and pleasant picture of the first singular vectors and the first singular value. Note that the greatest stretch occurs in two opposite directions, reflecting the $+/-$ ambiguity of singular vectors.

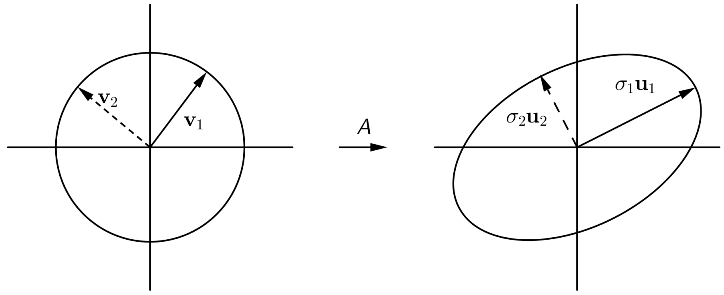

Figure: For 2×2 matrices $\mathbf{u}_1$ lies in the direction of greatest stretch, and $\mathbf{v}_1$ is the pre-image of $\sigma_1\mathbf{u}_1$

Since the second singular vector $\mathbf{v}_2$ is known to be a unit vector orthogonal to $\mathbf{v}_1$, its dotted-in position on the left side of the figure is clear, and similarly $\mathbf{u}_2$ is the unit vector orthogonal to $\mathbf{u}_1$, and so it points in the direction of the dotted vector on the right coordinate system. Furthermore, σ_2 is the length of the shortest semi-axis of the ellipse; equivalently, it can be found by a "least stretch" exploration similar to that for the "greatest stretch."

5. Sample uses of the interactive GUI

The primary use that the GUI serves is to function as a tool allowing users to input a 2×2 matrix of their choice and to visually watch the shape of the ellipse emerge under their control by their manipulation of the slide bar. Subsequently the user can further manipulate the slide bar to close in on the direction of greatest stretch, thereby revealing the numeric values of the first singular vectors and the first singular value. As noted earlier, the second singular vectors and value can also be visually located and numerically determined.

The second and equally important use of the GUI is to allow the user to develop an intuitive feel for predicting the shape of the ellipse for certain types of 2×2 matrices, and so equivalently to develop the ability to make a reasonable guess about its singular vectors and values. Considerable progress in developing this skill can be achieved by playing with the GUI for an hour or so. For example one can plug in numeric values for the matrix entries a, b, c, and d, and after noting the shape of the ellipse, then repeatedly doubling one of the entries (for instance a), each time observing through more slide-baring, the changes in the shape of the ellipse. In this way one begins to develop some intuition for how having a single relatively large individual entry affects the shape of the ellipse, and thereby the singular vectors and also the size of the singular value. Incidentally there is a shortcut to avoid typing in a new matrix each time - one can simply click on the entry box of the entry of interest, type in a new value and click the "calculate" button.

Other instructive activities for developing an intuitive feel for relating the shape of the ellipse to the size of the various matrix entries, include for instance, simultaneously increasing the size of a pair of entries, for example a and b, or a and d. Alternatively one can multiply (or divide) all entries by a factor of 10, or negate two (or all) entries (surprise).

6. Generalizations

The interactive GUI attached to this article works for real 2×2 matrices. The greatest stretch method could theoretically be used equally well to locate singular vectors for real 3×3 matrices [1], and even for $m\times n$ matrices if either m or n is 2 or 3. It is even valid for all real matrices; however, its elementary appeal evaporates if both m and n exceed 3.

In order to use the interactive GUI, you must first download it to your computer.

References

[1] John Clifford, David James, Michael Lachance, and Joan Remski, *A Constructive Approach to Singular Value Decomposition and Symmetric Schur Factorization*, Amer. Math. Monthly **112** (April, 2005), 358–363.

[2] David James, Michael Lachance, and Joan Remski, *Singular vectors' subtle secrets*, College Math. J. **42** (2011), no. 2, 86–95.

[3] D. Kalman, *A Singularly Valuable Decomposition: The SVD of a Matrix*, College Math. J. **26** (1996), no. 1, 2–23.

[4] Carla D. Martin and Mason A. Porter, *The extraordinary SVD*, Amer. Math. Monthly **119** (2012), no. 10, 838–851.

[5] Gilbert Strang, *Introduction to Linear Algebra, 5th Edition*, Wellesley-Cambridge Press, 2016.